LARGE DEVIATIONS FOR WEIGHTED SUMS OF STRETCHED EXPONENTIAL RANDOM VARIABLES

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Abstract. We consider the probability that a weighted sum of \( n \) i.i.d. random variables \( X_j, j = 1, \ldots, n \), with stretched exponential tails is larger than its expectation and determine the rate of its decay, under suitable conditions on the weights. We show that the decay is subexponential, and identify the rate function in terms of the tails of \( X_j \) and the weights. Our result generalizes the large deviation principle given by Kiesel and Stadtmüller [8] as well as the tail asymptotics for sums of i.i.d. random variables provided by Nagaev [10, 11]. As an application of our result, motivated by random projections of high-dimensional vectors, we consider the case of random, self-normalized weights that are independent of the sequence \( \{X_j\}_{j \in \mathbb{N}} \), identify the decay rate for both the quenched and annealed large deviations in this case, and show that they coincide. As another example we consider weights derived from kernel functions that arise in non-parametric regression.

1. Introduction

Let \( \{X_j\}_{j \in \mathbb{N}} \) be a sequence of independent and identically distributed (i.i.d.) random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with values in \( \mathbb{R} \) and with finite expectation \( m := \mathbb{E}[X_1] < \infty \). For \( n \in \mathbb{N} \), let \( S_n := \sum_{j=1}^{n} X_j \), denote the partial sum and \( \bar{S}_n := S_n / n \) the empirical mean values. The strong law of large numbers implies that \( \bar{S}_n \to m \) almost surely. Cramér’s Theorem on large deviations tells us that, if the \( X_j \) have finite exponential moments, that is, there exists \( t > 0 \) such that
\[
M(t) := \mathbb{E}[\exp(tX_1)] < \infty,
\]
then for any \( x > m \), the probability \( \mathbb{P}(\bar{S}_n \geq x) \) decays exponentially. More precisely,
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\bar{S}_n \geq x) = -\Lambda^*(x),
\]
where \( \Lambda^*(x) := \sup_{t \geq 0} \{tx - \log M(t)\} > 0 \). We will refer to this case as the “light-tailed” case. It is well known that if \( M(t) = +\infty \) for all \( t > 0 \), the probabilities \( P(\bar{S}_n \geq x) \) decay slower than exponentially. The reason is that, in contrast to when (1.1) holds, a “deviation” of the type \( \bar{S}_n \geq x \) is produced by the event that just one of the random variables takes a large value. For instance, if there is \( r \in (0, 1) \) and \( c > 0 \) such that \( P(X_1 \geq t) = c \exp(-t^r) \) for \( t \) large enough, then
\[
\lim_{n \to \infty} \frac{1}{n^r} \log \mathbb{P}(\bar{S}_n \geq x) = -(x - m)^r, \quad \forall x > m.
\]
The result in (1.2) goes back to [10] and it will also follow from our main result, Theorem 1. Cramér’s Theorem was generalized by [8] to weighted sums of i.i.d. random variables, see Section 2 below for a precise statement of their results. Our main result, Theorem 1, gives a corresponding statement for weighted sums of i.i.d. random variables with stretched exponential tails. One motivation to consider weighted sums, which is elaborated upon in Section 5.1, comes from random projections of high-dimensional vectors, which are of relevance in asymptotic geometric analysis [5, 9] and data analysis [2]. Another motivation stems from statistics (kernel functions, moving averages) considered for the light-tailed case in [8], since stretched exponential random variables arise in many applications. See Section 5.2 for an example.

This article is organized as follows: We first present the result and the regularity conditions from [8] in Section 2. Our main result, Theorem 1, is given in Section 3, and its proof is presented in Section 4. Finally, in Section 5.1, we give an application to random weights, and in Section 5.2, we consider weights derived from kernel functions that arise in non-parametric regression.

2. The Light-Tailed Case

For \( n \in \mathbb{N} \), let \( \{a_j(n)\}_{j \in \mathbb{N}} \) be a sequence of real numbers which we will call weights. For \( n \in \mathbb{N} \) define the weighted sum

\[
\bar{S}_n := \sum_{j=1}^{n} a_j(n) X_j
\]

and the measure \( \mu_n \) on \( B(\mathbb{R}) \), the set of Borel sets in \( \mathbb{R} \), as

\[
\mu_n(A) := P(\bar{S}_n \in A), \quad A \in B(\mathbb{R}).
\]

When the \( \{X_j\}_{j \in \mathbb{N}} \) have finite exponential moments, that is the moment generating function \( M(t) \) defined in (1.1) is finite for all \( t \in \mathbb{R} \), a large deviation principle for the sequence of weighted sums \( \{\bar{S}_n\}_{n \in \mathbb{N}} \) was established in [8] under suitable assumptions on the weights, see Assumption A below. The “classical” case of Cramér’s theorem corresponds to \( a_j(n) = 1/n, j = 1, 2, \ldots, n, n \in \mathbb{N} \).

Assumption A. (A.1) There exists a sequence of real numbers \( \{s_\nu\}_{\nu \in \mathbb{N}} \) such that \( s_\nu \neq 0 \) for all \( \nu \in \mathbb{N} \), the limit \( s := \lim_{\nu \to \infty} \sqrt[\nu]{|s_\nu|} \) exists and

\[
\sum_{j=1}^{n} a_j(n)^\nu = \frac{s_\nu}{n^{\nu-1}} R(\nu, n) \quad \text{for all } \nu \text{ and } n \in \mathbb{N},
\]

for some function \( R : \mathbb{N}^2 \to \mathbb{R} \) that satisfies, for every \( \nu \in \mathbb{N} \), \( R(\nu, n) \to 1 \) as \( n \to \infty \).

(A.2) There exist sequences \( \{r_\nu\}_{\nu \in \mathbb{N}} \) and \( \{\delta_\nu\}_{\nu \in \mathbb{N}} \) such that \( \limsup_{\nu \to \infty} \sqrt[\nu]{r_\nu} \leq 1 \), \( \lim_{n \to \infty} \delta_\nu = 0 \) and the error term satisfies

\[
|R(\nu, n) - 1| \leq r_\nu \left(1 + \frac{\delta_\nu}{n}\right)^\nu \quad \text{for all } \nu \text{ and } n.
\]

Now, let \( \Lambda \) denote the cumulant (or log moment) generating function of \( X_1 \), and let \( \{c_\nu\}_{\nu \in \mathbb{N}} \) be the sequence of coefficients that arise in the power series expansion for \( \Lambda \):

\[
\Lambda(t) := \log M(t) = \sum_{\nu=1}^{\infty} \frac{c_\nu}{\nu!} t^\nu, \quad t \in \mathbb{R}.
\]
Also, for \( t > 0 \), let \( \chi(t) := \sum_{\nu=1}^{\infty} \frac{\nu c_\nu}{\nu^\nu} t^\nu \), and let \( \chi^* \) denote its Legendre-Fenchel transform:

\[
\chi^*(t) := \sup_{t \in \mathbb{R}} \{tx - \chi(t)\}.
\]

It was shown in [8] that under Assumption A the sequence of measures \( \{\mu_n\}_{n \in \mathbb{N}} \) on \( B(\mathbb{R}) \) defined in (2.4) satisfies a large deviation principle with speed \( n \) and rate function \( \chi^* \). Recall that this means that

\[
- \inf_{x \in A^\circ} \chi^*(x) \leq \liminf_{n \to \infty} \frac{1}{n} \mu_n(A^\circ) \leq \limsup_{n \to \infty} \mu_n(A) \leq \inf_{x \in \bar{A}} \chi^*(x), \quad \forall A \in B(\mathbb{R}),
\]

where \( A^\circ \) and \( \bar{A} \), respectively, represent the interior and the closure of the set \( A \).

**Remark 2.1.** In fact, [8] provides a more general result that considers an infinite sum and refers to a general scale within the regularity conditions (cf. Assumption A), that is, they prove large deviations for the family of weighted sums of the form

\[
A(\lambda) := \sum_{j=1}^{\infty} a_j(\lambda) X_j,
\]

where \( \lambda \in I \) and either \( I = \mathbb{N} \) or \( I = [0, \infty) \).

Our goal will be to relax the finiteness assumption (2.7) on the moment generating function \( M(\cdot) \).

### 3. Main Result

In order to present our large deviation result for weighted sums of stretched exponential random variables, we will use slightly different assumptions on the weights from those used in [8]. We will restrict our considerations to non-negative weights. As we show in Lemma 3.3 below, in this case, our assumptions are weaker than those used in [8].

**Assumption B.** (B.1) There exists a real number \( s_1 \neq 0 \) such that the sequence \( \{R(1,n)\}_{n \in \mathbb{N}} \) of real numbers defined by

\[
\sum_{j=1}^{n} a_j(n) = s_1 R(1,n), \quad \text{for all } n \in \mathbb{N},
\]

satisfies \( R(1,n) \to 1 \) as \( n \to \infty \).

(B.2) There exists a real number \( s \) such that for \( a_{\max}(n) := \max_{1 \leq j \leq n} a_j(n) \),

\[
\lim_{n \to \infty} n \cdot a_{\max}(n) = s.
\]

Examples for weight sequences that satisfy both Assumption A and Assumption B include Valiron means, see [8] as well as kernel functions (see Section 5.2).

Recall that a function \( \ell : (0, \infty) \to (0, \infty) \) is called **slowly varying** (at infinity) if for every \( a > 0 \),

\[
\lim_{x \to \infty} \frac{\ell(ax)}{\ell(x)} = 1.
\]

We now state our main result.

**Theorem 1** (Large Deviations for Weighted Sums, Stretched Exponential Tails). Let \( \{X_j\}_{j \in \mathbb{N}} \) be a sequence of i.i.d. random variables on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with

\[
\mathbb{E}[X_j^k] < \infty \quad \forall k \in \mathbb{N},
\]

\[
3.11
\]

\[
\mathbb{E}[X^k] < \infty \quad \forall k \in \mathbb{N},
\]
and let \( m := \mathbb{E}[X_1] \). Suppose that there exist a constant \( r \in (0, 1) \) and slowly varying functions \( b, c_1, c_2 : (0, \infty) \to (0, \infty) \) and a constant \( t^* > 0 \) such that for \( t \geq t^* \),

\[
(3.12) \quad c_1(t) \exp (-b(t)t^r) \leq \mathbb{P} (X_1 \geq t) \leq c_2(t) \exp (-b(t)t^r).
\]

For every \( n \in \mathbb{N} \), let \( \{a_j(n)\}_{j \in \mathbb{N}} \) be a sequence of non-negative numbers that satisfy Assumption B with associated constants \( s_1, s_2 \in \mathbb{R} \), and let \( \{S_n\}_{n \in \mathbb{N}} \) be the sequence of weighted sums defined in (2.3). Then

\[
(3.13) \quad \lim_{n \to \infty} \frac{1}{b(n)n^r} \log \mathbb{P} (S_n \geq x) = - \left( \frac{x}{s} - \frac{s_1}{s_3} m \right)^r, \quad \forall x > s_1 m.
\]

**Remark 3.1.** The non-negativity assumption on the weights could be relaxed only if one had more information about the lower tail of the \( \{X_j\} \), that is, about the probabilitites \( \mathbb{P}(X_1 \leq -t) \) for \( t > 0 \). Consider the following example: \( a_j(n) = 1/n, j = 1, \ldots, [2n/3], a_j(n) = -1/n, j = [2n/3] + 1, \ldots, n \) (where, for \( z \in \mathbb{R} \), \( \lfloor z \rfloor \) represents the greatest integer less than or equal to \( z \)). Then Assumption B is satisfied with \( s_1 = 1/3 \) and \( s = 1 \). Take i.i.d. random variables \( \{X_j\}_{j \in \mathbb{N}} \) with mean \( m \) that satisfy (3.11) and (3.12) and, in addition, satisfy \( \mathbb{P}(X_1 \leq -t) = \exp(-t^{\alpha}) \) for some \( \alpha \) with \( 0 < \alpha < r \), and \( t \) large enough. Then, for every \( x > m/3 \), it can be shown that

\[
(3.14) \quad \lim_{n \to \infty} \frac{1}{n^\alpha} \log \mathbb{P} (S_n \geq x) = - \left( x - \frac{m}{3} \right)^\alpha.
\]

Indeed, to show (3.14), for any \( \varepsilon > 0 \), first write

\[
\mathbb{P}(S_n \geq x) \leq \mathbb{P} \left( \sum_{i=1}^{[2n/3]} X_i \geq 2n(m + \varepsilon)/3 \right) + \mathbb{P} \left( \sum_{i=[2n/3]+1}^{n} (-X_i) \geq n(x - 2(m + \varepsilon)/3) \right).
\]

Then, applying Theorem 1 twice, first to \( \{X_j\}_{j \in \mathbb{N}} \) and then to \( \{-X_j\}_{j \in \mathbb{N}} \), both times with \( a_j(n) = 1/n, j \in \mathbb{N} \), and recalling that \( \alpha < r \), we infer that as \( n \to \infty \), \( n^{-\alpha} \ln \mathbb{P}(\sum_{i=1}^{[2n/3]} X_i \geq 2n(m + \varepsilon)/3) = -\infty \) and hence,

\[
\limsup_{n \to \infty} \frac{1}{n^\alpha} \log \mathbb{P} (S_n \geq x) \leq \lim_{n \to \infty} \frac{1}{n^\alpha} \log \mathbb{P} \left( \sum_{i=[2n/3]+1}^{n} (-X_i) \geq n(x - 2((m + \varepsilon)/3) \right) = - \left( x - \frac{m - 2\varepsilon}{3} \right)^\alpha.
\]

Sending \( \varepsilon \to 0 \), we see that (3.14) holds with \( \leq \) instead of equality. To show the opposite inequality in (3.14), write

\[
\mathbb{P}(S_n \geq x) \geq \mathbb{P} \left( \sum_{i=1}^{[2n/3]} X_i \geq n(2m/3 - \varepsilon) \right) \cdot \mathbb{P} \left( \sum_{i=[2n/3]+1}^{n} (-X_i) \geq n(x - 2m/3 + \varepsilon) \right).
\]

The first probability on the right-hand side goes to 1 due to the law of large numbers. Once again, applying Theorem 1 to \( \{-X_j\}_{j \in \mathbb{N}} \) with \( a_j(n) = 1/n, j \in \mathbb{N} \), for the second term on the right-hand side, and then letting \( \varepsilon \to 0 \), we obtain (3.14) with \( \geq \) instead of equality. Together, both inequalities prove (3.14). However, we cannot recover \( \alpha \) from the assumptions in Theorem 1.
Remark 3.2. For the same reason as in the last remark, namely that the only assumption on the lower tail of \( \{X_j\}_{j \in \mathbb{N}} \) is (3.11), we cannot strengthen (3.12) to a large deviation principle without imposing further assumptions. For \( x < s_1m \), the decay of \( \mathbb{P}(\bar{S}_n \leq x) \) is determined by the lower tail of the \( \{X_j\} \). For example, if the \( \{X_j\}_{j \in \mathbb{N}} \) are bounded below, Cramér’s Theorem implies that \( \mathbb{P}(\bar{S}_n \leq x) \) decays exponentially in \( n \). If, on the other hand, \( \mathbb{P}(X_1 \leq -t) = \exp(-t^\alpha) \) with \( 0 < \alpha < r \), then as in Remark 3.1 we can show \(-\infty < \lim_{n \to \infty} n^{-\alpha} \log \mathbb{P}(\bar{S}_n \leq x) < 0 \).

Stretched exponential distributions have been proposed as a complement to the frequently used power law distributions to model many naturally occurring heavy-tailed distributions. Any distribution that satisfies (3.12) and is bounded below also satisfies (3.11). A concrete example is the Weibull distribution with shape parameter lying in the interval \( (0,1) \). Before proceeding to the proof of Theorem 1, let us comment on the relationship between Assumptions A and B. In fact, for a non-negative sequence of weights, Assumption B is weaker than Assumption A, see Lemma 3.3. To see that it is strictly weaker, consider the sequence of weights defined by \( a_j(n) = n^{-r} + n^{-1}(1 + \delta_j) \), \( j = 1, \ldots, n \), for some \( \varepsilon \in (0, \frac{1}{2}) \), for which it is easy to show that Assumption B holds, but (A.2) cannot be satisfied.

Lemma 3.3 (Relationship between Assumptions A and B). For every \( n \in \mathbb{N} \), let \( \{a_j(n)\}_{j \in \mathbb{N}} \) be a sequence of non-negative real numbers that satisfy Assumption A. Then Assumption B holds.

Proof. Given weights \( \{a_j(n)\}_{j \in \mathbb{N}} \) that satisfy Assumption A, clearly (B.1) follows immediately from (A.1). It only remains to show (B.2). First, note that by Assumption (A.2), \( R(\nu, n) \) satisfies the inequality

\[
1 - r_\nu \frac{(1 + \delta_n)^\nu}{n} \leq R(\nu, n) \leq 1 + r_\nu \frac{(1 + \delta_n)^\nu}{n}.
\]

Moreover, for any \( \varepsilon > 0 \), we can find \( \nu^*(\varepsilon) \in \mathbb{N} \) and \( n^*(\varepsilon) \in \mathbb{N} \) such that

\[
0 \leq r_\nu \leq (1 + \varepsilon)^\nu, \quad \forall \nu \geq \nu^*(\varepsilon), \quad \text{and} \quad 0 \leq \delta_n \leq \varepsilon, \quad \forall n \geq n^*(\varepsilon).
\]

By using the inequality \( a_{\max}(n)^\nu \leq \sum_{j=1}^{n} a_j(n)^\nu \), (A.1) and (A.2) we see that for \( \nu, n \in \mathbb{N} \),

\[
na_{\max}(n) \leq n \left( \sum_{j=1}^{n} a_j(n)^\nu \right)^{\frac{1}{\nu}} = n(s_\nu R(\nu, n))^{\frac{1}{\nu}} \cdot (n^{1-\nu})^{\frac{1}{\nu}} \leq n^{\frac{1}{\nu}}(s_\nu)^{\frac{1}{\nu}} \left( 1 + r_\nu \frac{(1 + \delta_n)^\nu}{n} \right)^{\frac{1}{\nu}}.
\]

Together with (3.16), this implies that for \( \varepsilon > 0 \), and \( \nu \geq \nu^*(\varepsilon), n \geq n^*(\varepsilon) \),

\[
na_{\max}(n) \leq (s_\nu)^{\frac{1}{\nu}} \left( n(1 + \varepsilon)^{2\nu} + (1 + \varepsilon)^{2\nu} \right)^{\frac{1}{\nu}} = (n + 1)^{\frac{1}{\nu}}(s_\nu)^{\frac{1}{\nu}}(1 + \varepsilon)^{2\nu}.
\]

Setting \( \nu = n \), for \( n \geq \max\{\nu^*(\varepsilon), n^*(\varepsilon)\} \), we have

\[
na_{\max}(n) \leq \sqrt{n} + 1 \sqrt{s_\nu}(1 + \varepsilon)^2.
\]

Since \( s = \lim_{n \to \infty} \sqrt[n]{s_\nu} \) by (A.1), taking first the limit superior as \( n \to \infty \) and then as \( \varepsilon \downarrow 0 \), we see that

\[
\limsup_{n \to \infty} na_{\max}(n) \leq \lim_{\varepsilon \downarrow 0} s(1 + \varepsilon)^2 = s.
\]
Next, for the lower bound for \( n_{a_{\max}}(n) \), we will make use of the fact that \( (n_{a_{\max}}(n))^\nu \geq n^{\nu-1} \sum_{j=1}^n a_j(n)^\nu \). Indeed, then for \( \varepsilon > 0 \), by (2.5), (2.6) and (3.16), for \( \nu \geq \nu^*(\varepsilon) \) and \( n \geq n^*(\varepsilon) \), we have

\[
na_{\max}(n) \geq (s_\nu R(\nu, n))^\frac{1}{\nu} \geq (s_\nu)^\frac{1}{\nu} \left( 1 - r_\nu \frac{(1 + \delta_n)^\nu}{n} \right)^{\frac{1}{\nu}} \geq (s_\nu)^\frac{1}{\nu} \left( 1 - \frac{(1+\varepsilon)^{2\nu}}{n} \right)^{\frac{1}{\nu}}.
\]

Taking limits as \( n \to \infty \) and noting that \( (1 - \frac{(1+\varepsilon)^{2\nu}}{n}) \sim \exp\{-(1 + \varepsilon)^{2\nu}\} \) and \( n\nu \to \infty \) as \( n \to \infty \), we obtain

\[
\liminf_{n \to \infty} na_{\max}(n) \geq (s_\nu)^{\frac{1}{\nu}} \liminf_{n \to \infty} \left( 1 - \frac{(1+\varepsilon)^{2\nu}}{n} \right)^{\frac{1}{\nu}} \geq (s_\nu)^{\frac{1}{\nu}}, \quad \forall \nu \geq \nu^*(\varepsilon).
\]

Sending \( \nu \to \infty \) and recalling from (A.1) that \( s = \lim_{\nu \to \infty} \sqrt[\nu]{s_\nu} \), we conclude that

\[
(3.18) \quad \liminf_{n \to \infty} na_{\max}(n) \geq s.
\]

Combining (3.17) and (3.18), we see that the weights \( \{a_j\}_{j \in \mathbb{N}} \) satisfy (B.2), and thus Assumption B. \( \square \)

4. Proof of Theorem 1

We will prove a slightly stronger statement than Theorem 1, namely we show in Section 4.2 that if the first inequality in (3.12) is satisfied, then the lower bound

\[
(4.19) \quad \liminf_{n \to \infty} \frac{1}{b(n)n^r} \log \mathbb{P} \left( S_n \geq x \right) \geq -\left( \frac{x}{s} - s_1 m \right)^r, \quad \forall x > s_1 m,
\]

holds; and in Section 4.3 we show that the second inequality in (3.12) implies the upper bound

\[
(4.20) \quad \limsup_{n \to \infty} \frac{1}{b(n)n^r} \log \mathbb{P} \left( S_n \geq x \right) \leq -\left( \frac{x}{s} - s_1 m \right)^r, \quad \forall x > s_1 m.
\]

First, in Section 4.1, we summarize some relevant properties of slowly varying functions. Throughout the section, the notation \( f(x) \sim g(x) \) as \( x \to \infty \) for two functions \( f, g : \mathbb{R} \to \mathbb{R} \) means that \( \lim_{x \to \infty} f(x)/g(x) = 1 \). Also, given a set \( A, 1_A \) will denote the indicator function of \( A \), which equals 1 on \( A \) and 0 on the complement.

4.1. Properties of Slowly Varying Functions. We will need the following preliminaries on slowly varying functions. Proposition 3 corresponds to Proposition 1.3.6 in [1], where Lemma 4 refers to (1.4) in [6].

**Proposition 4.1** (Properties of Slowly Varying Functions). Let \( \ell : (0, \infty) \to (0, \infty) \) be a slowly varying function (at infinity). Then

(i) \( \lim_{x \to \infty} \frac{\log \ell(x)}{\log x} = 0. \)

(ii) For any \( \alpha \in \mathbb{R} \), the function \( f(x) = \ell(x)^\alpha, x \in \mathbb{R} \), is slowly varying.

(iii) For any \( \alpha > 0 \), \( x^\alpha \ell(x) \to \infty \) and \( x^{-\alpha} \ell(x) \to 0 \) as \( x \to \infty \).

Furthermore, if \( m : (0, \infty) \to (0, \infty) \) is another slowly varying function then

(iv) the functions \( f(x) = \ell(x)m(x) \) and \( g(x) = \ell(x) + m(x), x \in \mathbb{R}, \) are slowly varying.

(v) if \( m(x) \to \infty \) as \( x \to \infty \), then the function \( f(x) = \ell(m(x)), x \in \mathbb{R}, \) is slowly varying.
Lemma 4.2 (Representation for Slowly Varying Functions). A function $\ell : (0, \infty) \to (0, \infty)$ is slowly varying if and only if there exist $a > 0$, $\eta \in \mathbb{R}$ and bounded measurable functions $\eta(\cdot)$ and $\varepsilon(\cdot)$ with $\eta(x) \to \eta$, $\varepsilon(x) \to 0$ as $x \to \infty$ such that, for $x \geq a$, $\ell$ can be written in the form

\[
\ell(x) = \exp \left\{ \eta(x) + \int_{a}^{x} \frac{\varepsilon(u)}{u} \, du \right\}.
\]

As a direct consequence of Lemma 4.2, we have the following result.

Lemma 4.3. Let $\ell : (0, \infty) \to (0, \infty)$ be a slowly varying function and let $g : (0, \infty) \to (0, \infty)$ be another function such that $g(x) \to c$ for some $c \in (0, \infty)$ as $x \to \infty$. Then we have

\[
\lim_{x \to \infty} \frac{\ell(g(x)x)}{\ell(x)} = 1.
\]

4.2. The Lower Bound. For $n \in \mathbb{N}$, let $j^*(n) := \inf\{1 \leq j \leq n : a_j(n) = a_{\max}(n)\}$. For any fixed $\varepsilon > 0$, since the $\{X_j\}_{j \in \mathbb{N}}$ are i.i.d.,

\[
\mathbb{P}(\bar{S}_n \geq x) = \mathbb{P} \left( \sum_{j=1}^{n} a_j(n)(X_j - m) \geq x - \sum_{j=1}^{n} a_j(n)m \right)
\]

\[
\geq \mathbb{P} \left( a_{\max}(n)(X_{j^*(n)} - m) \geq x - \sum_{j=1}^{n} a_j(n)m + \varepsilon, \sum_{j \in \{1, \ldots, n\}, j \neq j^*(n)} a_j(n)(X_j - m) \geq -\varepsilon \right)
\]

\[
= \mathbb{P}(X_1 \geq t_1(n)) \mathbb{P} \left( \sum_{j \in \{1, \ldots, n\}, j \neq j^*(n)} a_j(n)(X_j - m) \geq -\varepsilon \right),
\]

where $t_1(n) = t_1^*(n)$ is defined by

\[
t_1(n) := \frac{1}{na_{\max}(n)} \left[ n \left( x - \sum_{j=1}^{n} a_j(n)m + a_{\max}(n)m + \varepsilon \right) \right], \quad n \in \mathbb{N}.
\]

Applying the lower bound of (3.12) with $t = t_1(n)$, we obtain

\[
\mathbb{P}(\bar{S}_n \geq x) \geq c_1(t_1(n)) \exp \{-b(t_1(n))(t_1(n))^r\} \cdot \mathbb{P} \left( \sum_{j \in \{1, \ldots, n\}, j \neq j^*(n)} a_j(n)(X_j - m) \geq -\varepsilon \right).
\]

Note that by Assumption B, $t_1(n) \sim \left(\frac{x}{a} - \frac{a}{a_{\max}(n)m + \varepsilon} \right) n$ as $n \to \infty$. Since $c_1(\cdot)$ and $b(\cdot)$ are slowly varying functions, Lemma 4.3 implies that $c_1(t_1(n)) \sim c_1(n)$ and $b(t_1(n)) \sim b(n)$ as $n \to \infty$. Moreover, note that for some fixed $\delta \in (0, r)$, we can express

\[
\log c_1(n)/b(n)n^r = (\log c_1(n)/\log n)(\log n/n^\delta)(b(n)n^{r-\delta})^{-1},
\]

and the right-hand side goes to zero as $n \to \infty$ by properties (i) and (iii) of Proposition 4.1. Furthermore, since the $\{X_j\}$ have finite second moments by (3.11), and (B.2) implies that $\sum_{j=1,j \neq j^*(n)}^{n} a_j(n)^2 \leq n(a_{\max}(n))^2 \to 0$ as $n \to \infty$, it follows that $\sum_{j \in \{1, \ldots, n\}, j \neq j^*(n)} a_j(n)(X_j - m)$ converges to 0 in $L^2$. In turn, this implies that $\lim_{n \to \infty} \mathbb{P}(\sum_{j \in \{1, \ldots, n\}, j \neq j^*(n)} a_j(n)(X_j - m) \geq -\varepsilon) = 1$. Thus, taking logarithms of both sides of (4.24), then dividing by $b(n)n^r$ and sending first $n \to \infty$, and then $\varepsilon \downarrow 0$, we obtain the lower bound (4.19).
4.3. The Upper Bound. Let \( t_2(n) := n \left( \frac{x}{s} - \frac{s_1}{s} m \right) \). Then, we can write

\[
P \left( \bar{S}_n \geq x \right) \leq A_1^n + A_2^n,
\]

where, for \( n \in \mathbb{N} \),

\[
A_1^n := \mathbb{P} \left( \max_{1 \leq j \leq n} X_j \geq t_2(n) \right), \quad A_2^n := \mathbb{P} \left( \bar{S}_n \geq x, \max_{1 \leq j \leq n} X_j < t_2(n) \right).
\]

The union bound and the upper tail bound for \( X_1 \) in (3.12) imply that

\[
A_1^n \leq n \mathbb{P}(X_1 \geq t_2(n)) \leq nc_2(t_2(n)) \cdot \exp \{ -b(t_2(n)) (t_2(n))^r \}.
\]

Since \( b \) is slowly varying, \( b(t_2(n)) \sim b(n) \) as \( n \to \infty \), and properties (i) and (iii) of Proposition 4.1 show that \( \lim_{n \to \infty} \log n / b(n) n^r = \lim_{n \to \infty} \log c_2(t_2(n)) / b(n) n^r = 0 \). Together with the last display, this implies that

\[
\limsup_{n \to \infty} \frac{1}{b(n) n^r} \log A_1^n \leq \limsup_{n \to \infty} - \frac{(t_2(n))^r}{n^r} = - \left( \frac{x}{s} - \frac{s_1}{s} m \right)^r.
\]

Next, we turn to \( A_2^\beta \). Applying the exponential Chebyshev inequality with a positive real parameter \( \beta \) to (4.1) we obtain

\[
A_2^n \leq \exp \left\{ -\beta(n) \frac{x}{s} \right\} \cdot \prod_{j=1}^n \mathbb{E} \left[ \exp \left\{ \beta(n) \frac{a_j(n)}{s} X_j \right\} \cdot \mathbb{1}_{\{ X_j < t_2(n) \}} \right].
\]

Now, for \( \zeta > 0 \), define

\[
\beta(n) := \zeta n^r b \left( n \left( \frac{x}{s} - \frac{s_1}{s} m \right) \right) = \zeta n^r b(t_2(n)).
\]

Then, since \( b(\cdot) \) is slowly varying, \( \lim_{n \to \infty} \beta(n) / (b(n) n^r) = \zeta \). Together with (4.27) this implies that

\[
\limsup_{n \to \infty} \frac{1}{b(n) n^r} \log A_2^n \leq - \frac{x}{s} + \limsup_{n \to \infty} \frac{1}{b(n) n^r} \sum_{j=1}^n A_j^\beta(n),
\]

where, for \( j = 1, \ldots, n, n \in \mathbb{N}, \) and \( \zeta > 0 \), we define

\[
A_j^\beta(n) := \log \mathbb{E} \left[ \exp \left\{ \beta(n) \frac{a_j(n)}{s} X_j(n) \right\} \right], \quad \text{where } X_j(n) := X_j \mathbb{1}_{\{ X_j < t_2(n) \}}.
\]

We now show that the upper bound (4.20) is satisfied if the following proposition holds.

**Proposition 4.4** (Boundedness of the remainder). For every \( \zeta < (\frac{x}{s} - \frac{s_1}{s} m)^{r-1} \),

\[
\limsup_{n \to \infty} \frac{1}{b(n) n^r} \sum_{j=1}^n A_j^\beta(n) \leq \zeta m \frac{s_1}{s}.
\]

Indeed, given Proposition 4.4, we can substitute (4.31) into (4.29) and send \( \zeta \uparrow (\frac{x}{s} - \frac{s_1}{s} m)^{r-1} \) to conclude that

\[
\limsup_{n \to \infty} \frac{1}{b(n) n^r} \log A_2^n \leq - \left( \frac{x}{s} - \frac{s_1}{s} m \right)^r.
\]

Together with (4.25), and the analogous bound (4.26) for \( A_1^n \), we obtain the upper bound (4.20).

Thus, to prove the upper bound, it only remains to prove Proposition 4.4. We use similar techniques as in [7].
Proof of Proposition 4.4. Fix $\zeta < (\frac{1}{s} - \frac{a_i(n)}{s})^{-1}$ and denote $\beta(n)$ and $\Lambda^j$ simply as $\beta(n)$ and $\Lambda^j$. For the fixed $r \in (0, 1)$, we also choose $k \in \mathbb{N}$ such that $r < k/(k+1)$. Then, by the definition (4.30) of $\Lambda^j$, the estimates $\log x \leq x - 1$ for $x > 0$ and $e^x - 1 \leq x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \ldots + \frac{1}{(k+1)!} x^{k+1} e^x$, finiteness of the moments of $X_j$ due to (3.11), and the fact that $\beta(n)/(b(n)n^r) \to \zeta$ and $\sum_{j=1}^n a_j \to s_1$ as $n \to \infty$, we have

$$\limsup_{n \to \infty} \frac{1}{b(n)n^r} \sum_{j=1}^n \Lambda^j = \limsup_{n \to \infty} \frac{1}{b(n)n^r} \left( \sum_{j=1}^n \sum_{i=1}^k \frac{E \left( \beta(n) \frac{a_j(n)}{s} X_j(n)^i \right)}{i!} \right) + \frac{B_0}{(k+1)!},$$

with

$$B_0 := \limsup_{n \to \infty} \frac{1}{b(n)n^r} \sum_{j=1}^n \left( \beta(n) \frac{a_j(n)}{s} \right)^{k+1} \cdot \left[ (X_j(n)^{k+1} \exp \left( \beta(n) \frac{a_j(n)}{s} X_j(n) \right) \right].$$

Since, by Assumption B, \( \lim_{n \to \infty} \frac{1}{b(n)n^r} \sum_{j=1}^n E[(\beta(n) \frac{a_j(n)}{s} X_j(n))^i] = \zeta m_{\frac{a_i(n)}{s}} \) if $i = 1$, and is zero for $i \neq 1$, this implies

$$\limsup_{n \to \infty} \frac{1}{b(n)n^r} \sum_{j=1}^n \Lambda^j \leq \zeta m_{s_1} + \frac{B_0}{(k+1)!}.$$

To complete the proof of Proposition 4.4, it suffices to show that $B_0 = 0$. In this regard, we distinguish between the cases $X_j^{(n)} < t^*$ and $X_j^{(n)} \geq t^*$, where we recall that for $t \geq t^*$, (3.12) is satisfied. Specifically, we bound $B_0$ by $\limsup \{B_1(n) + B_2(n)\}$, where

(4.32)

$$B_1(n) := \frac{1}{(k+1)!} \frac{1}{b(n)n^r} \sum_{j=1}^n \left( \beta(n) \frac{a_j(n)}{s} \right)^{k+1} \cdot (t^*)^{k+1} \exp \left( \beta(n) \frac{a_j(n)}{s} t^* \right),$$

(4.33)

$$B_2(n) := \frac{1}{(k+1)!} \frac{1}{b(n)n^r} \sum_{j=1}^n \left( \beta(n) \frac{a_j(n)}{s} \right)^{k+1} \cdot E \left[ \left( X_j(n) \right)^{k+1} \exp \left( \beta(n) \frac{a_j(n)}{s} X_j(n) \right) \right] \cdot 1 \{ X_j^{(n)} \geq t^* \}.$$

We now show that both $B_1(n)$ and $B_2(n)$ converge to 0 as $n \to \infty$. Note that (B.2), the definition of $\beta(n)$ in (4.28) and, recalling $r < k/(k+1)$, property (iii) of Proposition 4.1 imply that

(4.34)

$$\lim_{n \to \infty} n \left( \beta(n) \frac{a_{\max}(n)}{s} \right)^{k+1} = \lim_{n \to \infty} \left( \frac{a_{\max}(n)n}{s} \right)^{k+1} \left( \zeta n^{r - \frac{k+1}{k+1}} b(n) \right)^{k+1} = 0,$$

and

(4.35)

$$\lim_{n \to \infty} \left( \beta(n) \frac{a_{\max}(n)}{s} \right) = 0.$$

Combined with (4.32) and recalling that $a_{\max}(n) := \max_{1 \leq j \leq n} a_j(n)$, this shows that $B_1(n) \to 0$ as $n \to \infty$. 
Now, to bound $B_2(n)$, first note that by Hölder’s inequality, for any $\varepsilon > 0$ we have
\[
\mathbb{E} \left[ (X_1^{(n)})^{k+1} \exp \left( \beta(n) \frac{a_{\max}(n)}{s} X_1^{(n)} \right) \mathbb{1}_{\{X_1^{(n)} \geq t^*\}} \right] 
\]
(4.36)
\[
\leq \mathbb{E} \left[ (X_1^{(n)})^{(k+1) \frac{1+\varepsilon}{1-\varepsilon}} \mathbb{1}_{\{X_1^{(n)} \geq t^*\}} \right]^{\frac{1}{1-\varepsilon}} \cdot \mathbb{E} \left[ \exp \left( (1+\varepsilon) \beta(n) \frac{a_{\max}(n)}{s} X_1^{(n)} \right) \mathbb{1}_{\{X_1^{(n)} \geq t^*\}} \right]^{\frac{1}{1+\varepsilon}}. 
\]

Due to the finiteness of the moments of $X_1$ assumed in (3.11), the limit in (4.34) yields
\[
\limsup_{n \to \infty} n \cdot \left( \beta(n) \frac{a_{\max}(n)}{s} \right)^{k+1} \mathbb{E} \left[ (X_1^{(n)})^{(k+1) \frac{1+\varepsilon}{1-\varepsilon}} \mathbb{1}_{\{X_1^{(n)} \geq t^*\}} \right]^{\frac{1}{1-\varepsilon}} = 0. 
\]
When combined with (4.33) and (4.36), to prove the convergence of $B_2(n)$ to zero, it clearly suffices to show that
\[
\limsup_{n \to \infty} \frac{1}{b(n)n^r} \mathbb{E} \left[ \exp \left( (1+\varepsilon) \beta(n) \frac{a_{\max}(n)}{s} X_1^{(n)} \right) \mathbb{1}_{\{X_1^{(n)} \geq t^*\}} \right]^{\frac{1}{1+\varepsilon}} < \infty
\]
for $\zeta < (1+\varepsilon)^{-1} \left( \frac{\varepsilon}{s} - \frac{a_{\max}(n)}{s} \right)^{r-1}$ and the claim follows as $\varepsilon \to 0$. To derive an upper bound for the expectation in (4.37) we will use the following integration-by-parts formula.

**Lemma 4.5** (Integration by parts). For any random variable $X$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and any $\alpha > 0$, $a, b \in \mathbb{R}$ with $a < b$ the following relation holds:
\[
\mathbb{E} \left[ \exp (\alpha X) \mathbb{1}_{\{a \leq X \leq b\}} \right] = \alpha \int_a^b \exp (\alpha z) \mathbb{P}(X \geq z) \, dz + \exp (\alpha a) \mathbb{P}(X \geq a) - \exp (\alpha b) \mathbb{P}(X > b). 
\]

Recalling that $X_j^{(n)} = X_j \mathbb{1}_{\{X_j < t_2(n)\}}$, applying Lemma 4.5 with $a = t^*$ and $b = t_2(n)$, we deduce that
\[
\frac{1}{b(n)n^r} \mathbb{E} \left[ \exp \left( (1+\varepsilon) \beta(n) \frac{a_{\max}(n)}{s} X_1^{(n)} \right) \mathbb{1}_{\{X_1^{(n)} \geq t^*\}} \right] 
\]
\[
\leq \frac{1}{b(n)n^r} \int_{t_2(n)}^{t^*} (1+\varepsilon) \beta(n) \frac{a_{\max}(n)}{s} \exp \left( (1+\varepsilon) \beta(n) \frac{a_{\max}(n)}{s} z \right) \mathbb{P}(X_1 \geq z) \, dz 
\]
\[
+ \frac{1}{b(n)n^r} \exp \left( (1+\varepsilon) \beta(n) \frac{a_{\max}(n)}{s} t^* \right). 
\]

Since $b(n)n^r \to \infty$, the second term on the right-hand side of (4.38) converges to 0 by (4.35).

Now, let $\zeta^* := \zeta \cdot \left( \frac{\varepsilon}{s} - \frac{a_{\max}(n)}{s} \right)$. Inserting the upper bound (3.12) on the tail of $X_1$, substituting $y := (t_2(n))^{-1}z$ and recalling the definition of $\beta(n)$ from (4.28), we see that the first term on the right-hand side of (4.38) is bounded above by
\[
(1+\varepsilon)\zeta^* \frac{b(t_2(n))}{b(n)} \frac{n a_{\max}(n)}{s} \int_{t_2(n)}^{t^*} I_n(y) \, dy,
\]
(4.39)
where the integrand $I_n(\cdot)$ is given by

$$I_n(y) := c_2(t_2(n)y) \exp \left\{ n^r b(t_2(n)) \left( (1 + \varepsilon) \zeta^* \frac{na_{\max}(n)}{s} y - \frac{b(t_2(n)y)}{b(t_2(n))} \left( \frac{x}{s} - \frac{s_1}{s}m \right) r y \right) \right\},$$

for $y \in (0, 1]$. Since $b(\cdot)$ is slowly varying and condition (B.2) holds, we see that the coefficient in front of the integral in (4.39) converges to $(1 + \varepsilon) \zeta^*$ as $n \to \infty$. It now remains to show that, for every $\zeta^* < (1 + \varepsilon)^{-1} \left( \frac{x}{s} - \frac{s_1}{s}m \right) r$, the integral in (4.39) stays bounded as $n \to \infty$. By the assumption that $b(\cdot)$ is slowly varying and since $r < 1$, for any fixed $y \in (0, 1]$ and any $\zeta^* < (1 + \varepsilon)^{-1} \left( \frac{x}{s} - \frac{s_1}{s}m \right) r$, it follows that $I_n(y) \to 0$ as $n \to \infty$. Therefore, we need to examine the lower limit of integration $y_n := t^*/(t_2(n))$ and show that $I_n(y_n)$ stays bounded as $n \to \infty$. Recalling that $t_2(n) = n(\frac{x}{s} - \frac{s_1}{s}m)$ and $\zeta^* = \zeta(\frac{x}{s} - \frac{s_1}{s}m)$, note that

$$I_n(y_n) = c_2(t^*) \exp \left\{ n^{r-1} b(t_2(n))(1 + \varepsilon) \zeta^* \frac{na_{\max}(n)}{s} t^* - b(t^*)(t^*)r \right\}.$$

Since $na_{\max}(n) \sim s$, $b(t_2(n)) \sim b(n)$ and $n^{r-1}b(n) \to 0$ as $n \to \infty$, it follows that $\limsup_{n \to \infty} I_n(y_n)$ is finite.

Thus, we have shown that $B^*_2 \to 0$ as $n \to \infty$ and hence, that $B_0 = 0$. This completes the proof of Proposition 4.4, and hence, the upper bound (4.20) and Theorem 1 follow.

$$\square$$

5. Examples

5.1. Example 1: Random Weights. We consider a sequence of strictly positive i.i.d. random variables $\{\theta_j\}_{j \in \mathbb{N}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that they are $\mathbb{P}$-almost surely uniformly bounded, that is, their essential supremum is finite:

$$(5.40) \quad M^* := \inf \{ a \in \mathbb{R} : \mathbb{P}(\theta_1 > a) = 0 \} < \infty.$$ 

Furthermore, define the triangular array of weights $\{a_j(n, \theta_1, \ldots, \theta_n), j = 1, \ldots, n\}_{n \in \mathbb{N}}$ by

$$(5.41) \quad a_j(n, \theta_1, \ldots, \theta_n) := \frac{\theta_j}{\sum_{i=1}^n \theta_i}, \quad j = 1, \ldots, n, n \in \mathbb{N},$$

and let $\{\bar{S}_n\}_{n \in \mathbb{N}}$ be the corresponding sequence of weighted sums:

$$(5.42) \quad \bar{S}_n := \sum_{j=1}^n a_j(n, \theta_1, \ldots, \theta_n) X_j = \sum_{j=1}^n \frac{\theta_j}{\sum_{i=1}^n \theta_i} X_j, \quad n \in \mathbb{N}.$$ 

We prove a large deviation theorem for the sequence of random weighted sums $\{\bar{S}_n\}_{n \in \mathbb{N}}$, both in the “quenched” (i.e., conditioned on the weight sequence $\{\theta_j\}_{j \in \mathbb{N}}$), and “annealed” (i.e., averaged over the weight sequence) cases. Note that $\bar{S}_n$ can be viewed as a random projection of the data $\{X_j\}$. Random projections have attracted much interest in recent research in applied mathematics as an important tool in data analysis and dimensionality reduction [2], as well as in asymptotic geometric analysis [5, 9].

**Theorem 2** (Large Deviations for Random Weights, Stretched Exponential Tails). Let $\{X_j\}_{j \in \mathbb{N}}$ be a sequence of i.i.d. random variables such as in Theorem 1 and let $\{\theta_j\}_{j \in \mathbb{N}}$ be a sequence of i.i.d. random variables which is independent of the sequence $\{X_j\}_{j \in \mathbb{N}},$ and
is almost surely uniformly bounded by $M^*$ as specified in (5.40). Define $S_n$ by (5.42). Then, for $x > m$, we have
\begin{equation}
\lim_{n \to \infty} \frac{1}{b(n)n^r} \log P(S_n \geq x | \theta_1, \theta_2, \ldots) = - \left[ \left( \frac{[\theta_1]}{M^*} \right) (x-m) \right]^r \ P\text{-a.s.,}
\end{equation}
and
\begin{equation}
\lim_{n \to \infty} \frac{1}{b(n)n^r} \log P(S_n \geq x) = - \left[ \left( \frac{[\theta_1]}{M^*} \right) (x-m) \right]^r .
\end{equation}

Proof. The proof of (5.43) is a direct application of Theorem 1. First of all, note that for every $n \in \mathbb{N}$, $\sum_{j=1}^n a_j(n, \theta_1, \ldots, \theta_n) = 1$ almost surely, and hence $s_1 = 1$, where $s_1$ is the quantity defined in (B.1). Furthermore,
\begin{equation}
n \cdot a_{\max}(n, \theta_1, \ldots, \theta_n) = n \cdot \max \{ \theta_j : 1 \leq j \leq n \} = \frac{\sum_{i=1}^n \theta_i}{\frac{1}{n} \sum_{i=1}^n \theta_i}.
\end{equation}

It is easy to check that almost surely, $\max \{ \theta_j : 1 \leq j \leq n \} \to M^*$ as $n \to \infty$. By the strong law of large numbers, it follows that almost surely, $n \cdot a_{\max}(n, \theta_1, \ldots, \theta_n) \to s := M^*/[\theta_1]$ as $n \to \infty$. By Theorem 1 we conclude that, for $x > m$, the quenched asymptotics (5.43) are valid.

We now turn to the proof of (5.44). Note that we have
\begin{equation}
P(S_n \geq x) = P \left( \frac{1}{n} \sum_{j=1}^n \theta_j X_j \geq \frac{\sum_{i=1}^n \theta_i}{\frac{1}{n} \sum_{i=1}^n \theta_i} x \right).
\end{equation}

Now, $\frac{1}{n} \sum_{i=1}^n \theta_i \to [\theta_1]$, $P$-almost surely, and the probability of a deviation decays exponentially in $n$, due to Cramér’s Theorem (recall that the $\{\theta_i\}$ are uniformly bounded!). We will now show that
\begin{equation}
\lim_{n \to \infty} \frac{1}{b(n)n^r} \log P(S_n \geq x) \approx \lim_{n \to \infty} \frac{1}{b(n)n^r} \log P \left( \frac{1}{n} \sum_{j=1}^n \theta_j X_j \geq [\theta_1] x \right),
\end{equation}
in the sense explained in (5.48) and (5.49) below. Fix $\delta > 0$ and consider the events $F_n := \{ \frac{1}{n} \sum_{i=1}^n \theta_i \geq (1 - \delta)[\theta_1] \}$ and their complements $F_n^c$ for $n \in \mathbb{N}$. Then, $P(S_n \geq x) \leq P(\frac{1}{n} \sum_{j=1}^n \theta_j X_j \geq (1 - \delta)[\theta_1] x) + P(F_n^c)$, and since $P(F_n^c)$ decays exponentially in $n$, it follows that for any $\delta > 0$,
\begin{equation}
\lim_{n \to \infty} \sup \frac{1}{b(n)n^r} \log P(S_n \geq x) \leq \lim_{n \to \infty} \sup \frac{1}{b(n)n^r} \log P \left( \frac{1}{n} \sum_{j=1}^n \theta_j X_j \geq (1 - \delta)[\theta_1] x \right).
\end{equation}

On the other hand, with $G_n := \{ \frac{1}{n} \sum_{i=1}^n \theta_i \leq (1 + \delta)[\theta_1] \}$, we have $P(S_n \geq x) \geq P(\{S_n \geq x\} \cap G_n) \geq P(\frac{1}{n} \sum_{j=1}^n \theta_j X_j \geq (1 + \delta)[\theta_1] x) - P(G_n^c)$, and since $P(G_n^c)$ decays exponentially in $n$, we have
\begin{equation}
\lim_{n \to \infty} \inf \frac{1}{b(n)n^r} \log P(S_n \geq x) \geq \lim_{n \to \infty} \inf \frac{1}{b(n)n^r} \log P \left( \frac{1}{n} \sum_{j=1}^n \theta_j X_j \geq (1 + \delta)[\theta_1] x \right).
\end{equation}
Proof. The proof is a direct application of Theorem 1 with i.i.d. random variables \( \theta_j X_j \) and weights \( a_j(n) = \frac{1}{n}, j = 1, \ldots, n \) that clearly satisfy Assumption B with \( s = s_1 = 1 \) and \( R(\nu, 1) = 1 \) for all \( \nu \in \mathbb{N} \). Considering the tail of \( \theta_1 X_1 \), we see that due to (3.12), for \( t \geq t^* \), \( \mathbb{P}(\theta_1 X_1 \geq t) \leq \mathbb{P}(X_1 \geq t/M^*) \leq c_2(t/M^*) \exp(-b(t/M^*)t^r(M^*)^{-r}) \).

On the other hand, for \( t \geq t^* \), again by (3.12), \( \mathbb{P}(\theta_1 X_1 \geq t) \geq \mathbb{P}(\theta_1 \geq M^* - \delta) \mathbb{P}(X_1 \geq t/(M^* - \delta)) \geq \mathbb{P}(\theta_1 \geq M^* - \delta) c_1(t/(M^* - \delta)) \exp(-b(t/(M^* - \delta))t^r(M^* - \delta)^{-r}) \). The proof is completed by applying the lower and upper bounds in (4.19) and (4.20), respectively, and then sending \( \delta \downarrow 0 \) to obtain (5.44).

\[ \square \]

**Remark 5.1.** The equality of the quenched and annealed rate functions in (5.43) and (5.44), respectively, is characteristic of our regime; it is in sharp contrast to the case of light-tailed random variables \( X_j \), that is, random variables \( X_j \) satisfying (1.1). In the light-tailed case, \( \mathbb{P}(S_n \geq x|\theta_1, \theta_2, \ldots) \) and \( \mathbb{P}(S_n \geq x) \) both decay exponentially in \( n \), but the rate functions will in general not be the same. This was one of the motivations for the present paper, and will be treated in forthcoming work.

### 5.2. Example 2: Kernel Functions

In non-parametric regression kernels are frequently used as weighting functions. They are an important tool to smooth data. Applications include the approximation of probability density functions and conditional expectations.

**Definition 5.2 (Kernel).** A kernel is an integrable function \( k : [-1, 1] \rightarrow [0, \infty) \) satisfying the following two requirements:

- (i) \( \int_{-1}^{1} k(u)du = 1 \).
- (ii) \( k(-u) = k(u) \quad \forall u \in [0, 1] \).

Define the triangular array of weights \( \{a_j(n), j = 1, \ldots, n\}_{n \in \mathbb{N}} \) by

\[
(5.50) \quad a_j(n) := \frac{1}{n} \cdot k \left( 2 \cdot \frac{j - n/2}{n} \right), \quad j = 1, \ldots, n, n \in \mathbb{N},
\]

and let \( \{S_n\}_{n \in \mathbb{N}} \) be the corresponding sequence of weighted sums:

\[
(5.51) \quad S_n := \sum_{j=1}^{n} a_j(n)X_j = \frac{1}{n} \sum_{j=1}^{n} k \left( 2 \cdot \frac{j - n/2}{n} \right) X_j, \quad n \in \mathbb{N}.
\]

**Theorem 3 (Large Deviations for Kernel Weighted Sums, Stretched Exponential Tails).** Let \( \{X_j\}_{j \in \mathbb{N}} \) be a sequence of i.i.d. random variables such as in Theorem 1 and let \( k : [-1, 1] \rightarrow [0, \infty) \) be a kernel. Define \( S_n \) by (5.51). Then, for \( x > m \), we have

\[
(5.52) \quad \lim_{n \rightarrow \infty} \frac{1}{b(n)n^r} \log \mathbb{P}(S_n \geq x) = - \left( \sup_{x \in [-1, 1]} k(x) \right)^{-r} (x - m)^r.
\]

**Proof.** The proof is a direct application of Theorem 1. Recall the definition of the quantities \( \{s_\nu\}_{\nu \in \mathbb{N}} \) from Assumption B. It is straightforward to check that \( s_\nu = \int_{-1}^{1} k^\nu(u)du \) (in particular \( s_1 = 1 \)). Therefore,

\[
(5.53) \quad s = \lim_{\nu \rightarrow \infty} \left( \int_{-1}^{1} k^\nu(u)du \right)^{1/\nu}.
\]
Since the $p$-norm converges to the supremum norm as $p \to \infty$, we conclude that $s = \sup_{x \in [-1,1]} k(x)$.

**Acknowledgments.** N. Gantert and F. Rembart thank the Division of Applied Mathematics, Brown University, Providence, for its hospitality. N. Gantert further thanks ICERM, Providence, for an invitation to the program “Computational Challenges in Probability” where this work was initiated.

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