Stochastic Analysis

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1 SRW

What happens if 1 scale SRW as follows divide time by n and space by $\sqrt{(n)}$? We will get some continious time process $B(t)$, $t \in \mathbb{R}^+$, and $B(t) \sim \mathcal{N}(0,t)$. It will have other properties

1. $S_l - S_k$ is indep. of S_k
2. take $0 < k_0 < k_1 < k_2 < ... < k_m$ then $S_{k_0}, S_{k_1} - S_{k_0}, S_{k_2} - S_{k_1}, ..., S_{k_n} - S_{k_{n-1}}$ are all independent
3. if $t_1 < t_2 < ... < t_m$, then $S_{t_1}, S_{t_2} - S_{t_1}, ..., S_{t_m} - S_{t_{m-1}}$ are all independent
4. if $t > \ S$ then $B_t - B_s \sim \mathcal{N}(0,1)$
5. continuity

2 Multivariate-Gaussian variables

2.1 Definition

*Let X be an n- dimensional random variable $X = (X_1, ..., X_n)$. We say that X is a **multivariate Gaussian variable** (MGW) if for every vector $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$,*

$$\langle X, \lambda \rangle := \sum_{k=1}^{n} \lambda_k \cdot X_k$$

has a normal distribution.

$X_1, X_2 \sim \mathcal{N}(0,1)$ independent. $X = (X_1, X_2)$ is a MGV. Why? We need to verify that for every choice of λ_1 and λ_2 , $\lambda_1 X_1 + \lambda_2 X_2$ is normally distributed. How do we verify this?

$\lambda_1 X_1 \sim \mathcal{N}(0, \lambda_2^2)$. $\lambda_2 X_2 \sim \mathcal{N}(0, \lambda_2^2) \ \rightarrow$

$$\varphi_{\lambda_1, X_1}(t) = \exp\left(-\frac{\lambda_1^2 t^2}{2}\right)$$

$$\varphi_{\lambda_2, X_2}(t) = \exp\left(-\frac{\lambda_2^2 t^2}{2}\right)$$

$\lambda_1 X_1$ and $\lambda_2 X_2$ are independent. With that

$$\begin{aligned}\varphi_{\lambda_1 X_1 + \lambda_2 X_2}(t) = \varphi_{\lambda_1 X_1}(t) \cdot \varphi_{\lambda_2 X_2} &= \exp\left(-\frac{\lambda_1^2 t^2}{2}\right) \cdot \exp\left(-\frac{\lambda_2^2 t^2}{2}\right) \\ &= \exp\left(-\frac{(\lambda_1^2 + \lambda_2^2) t^2}{2}\right)\end{aligned}$$

and this is the characteristic function of $\mathcal{N}(0, \lambda_1^2 + \lambda_2^2)$. Remembering that the characteristic function determines the distribution, we get that

$$\lambda_1 X_1 + \lambda_2 X_2 \sim \mathcal{N}(0, \lambda_1^2 + \lambda_2^2)$$

Let X be an N-dimensional random variable (RV). The **covariance matrix** Σ of X is the matrix s.t. $\Sigma_{ij} = cov(X_i, X_j)$ (provided that it is well defined). The expectation of X is the n-dim. vector $(\mathrm{E}(X_1), \mathrm{E}(X_2), ..., \mathrm{E}(X_n))$ (provided expectations exist). Theorem (will be proven in exercise section): The distribution of a MGW is determined by its expectation and its covariance matrix.

Important exercise
Let X_1, X_2 be independent $\mathcal{N}(0,1)$ variables. Let

$$Y_1 = \frac{X_1 + X_2}{\sqrt{2}} \qquad Y_2 = \frac{X_1 - X_2}{\sqrt{2}}$$

Then Y_1 and Y_2 are independent $\mathcal{N}(0,1)$ variables.

Let $X = (X_1, X_2)$ a two dimensional variable. What is the density $f_X : \mathbb{R}^2 \to \mathbb{R}$ of X?

$$\begin{aligned} f_X(x_1, x_2) &= f_{X_1}(x_1) \cdot f_{X_2}(x_2) \\ &= \frac{1}{2\pi} e^{-\frac{x_1^2 + x_2^2}{2}} \\ &= \frac{1}{2\pi} e^{-0.5r^2} \end{aligned}$$

where r is the distance of x_1, x_2 from origin consider the linear transformation

$$(x_1, x_2) \to \left(y_1 = \frac{x_1 + x_2}{\sqrt{2}}, y_2 = \frac{x_1 - x_2}{\sqrt{2}} \right)$$

This is a rotation by 45°. With this fact

$$\Rightarrow \quad y_1^2 + y_2^2 = x_1^2 + x_2^2$$

Now we conclude, that f is invariate under the rotation. (Y_1, Y_2) has the same density function as (X_1, X_2). Then, (Y_1, Y_2) has exact the same distribution as (X_1, X_2). Now we are done because X_1, X_2 are independent $\mathcal{N}(0,1)$.

3 Construction of the BM

We begin with BM running from time 0 to time 1. Let $\Omega = C[0,1]$ (space of continuous functions $[0,1] \to \mathbb{R}$). Let d be the supremum metrics $d(f,g) = \sup_{0 \le t \le 1} |f(t) - g(t)|$. Let $\mathcal{F}$ be the σ-algebra determined by this metric.

Wiener measure (distribution of BM) P is a probability measure on $(\Omega, \mathcal{F})$ such that.

3.1 Definition

Let $\Omega = C[0,1]$ with supremum metric

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

let $\mathcal{B}$ be the Borel $\sigma-$Algebra of this metric. A probability measure P on $(\Omega, \mathcal{B})$ is a ***Wiener measure*** *if*

1. $P\left(f(0)=0\right)=1$

2. for every $0\leq t_1<t_2<...<t_k\leq 1$*, the RV*

$$f(t_2)-f(t_1), f(t_3)-f(t_2), ..., f(t_k)-f(t_{k-1})$$

are independent.

3. for $t>s$ *,* $f(t)-f(s)\sim\mathcal{N}(0,t-s)$

3.2 Theorem

A Wiener measure exists and is unique. Therefore we speak of the Wiener measure.

Terminology: A function distributed according to the Wiener measure is called a Brownian Motion.

Proof of existence: For every dyadic number

$$v=\frac{k}{2^n},\ \ 0\leq k\leq 2^n\ ,\ n=0,1,...$$

Let $Z_v\sim\mathcal{N}(0,1)$ and assume (Z_v) is an iid. collection living on a prob space $(X,\mathcal{F},\mathrm{P})$.

For each $n=0,1,...$ we construct a random continuous function $f_n:[0,1]\to\mathbb{R}$. We define measurable function: $\phi_n:X\to\Omega$.

$f_0(0)=0,\ f_0(0)=Z_1$ linearly interpolate between 0 and 1.

Fix $C_n=\frac{1}{\sqrt{2^{n+1}}}$. Given f_{n-1}, we construct f_n as follows:

For every k even, take

$$f_n\left(\frac{k}{2^n}\right):=f_{n-1}\left(\frac{k}{2^n}\right)$$

For every k odd, take

$$f_n\left(\frac{k}{2^n}\right):=f_{n-1}\left(\frac{k}{2^n}\right)+\frac{Z_k}{2^n}\cdot C_n$$

For all other values linearly interpolate, i.e. $\frac{k}{2^n}<x\leq\frac{k+1}{2^n}$ then

$$f(x)=2^n\left[f\left(\frac{k}{2^n}\right)\cdot\left(\frac{k+1}{2^n}-x\right)+f\left(\frac{k+1}{2^n}\right)\cdot\left(x-\frac{k}{2^n}\right)\right]$$

Construct f_1:

$$f_0(0), f_0(1)=Z_1, f_0\left(\frac{1}{2}\right)=\frac{Z_1}{2}$$

$$f_1(0)=0, f_1(1)=f_1\left(\frac{z}{2^1}\right)=f_0(1)=Z_1,$$

$$f_1\left(\frac{1}{2}\right)=f_1\left(\frac{1}{2^1}\right)=f_0\left(\frac{1}{2}\right)-Z_{\frac{1}{2}}\cdot C_1=\frac{Z_1}{2}+\frac{Z_{\frac{1}{2}}}{2}$$

For all other points, linearly interpolate.

$$f_1(1)=Z_1,\quad f_1\left(\frac{1}{2}\right)=\frac{Z_1+Z_{\frac{1}{2}}}{2}$$

$$\Rightarrow f_1(1) - f_1\left(\frac{1}{2}\right) = \frac{Z_1 - Z_{\frac{1}{2}}}{2}, \quad f_1\left(\frac{1}{2}\right) = \frac{Z_1 + Z_{\frac{1}{2}}}{2}$$

$$\Rightarrow f_1(1) - f_1\left(\frac{1}{2}\right), \quad f_1\left(\frac{1}{2}\right) - f_1(0)$$

are independent and both are distributed $\mathcal{N}(0, \frac{1}{2})$. ■

3.3 Claim

With prob. 1, the sequence f_n converges uniformly. Let δ be the Limit, f is continious, f is BM.

[formally: Let $\phi: \ \Omega \Rightarrow \ \Omega$ be the function $\phi(w) = \lim \phi_n(w)$. Then the measure μ and Ω defined by

$$\mu(A) = \mathrm{P}\left(\phi^{-1}(A)\right)$$

is a Wiener measure.]

Proof: For every n, we estimate

$$\mathrm{P}\left(\sup_{x\in[0,1]} |f_n(x) - f_{n-1}(x)| > \frac{1}{n^2}\right)$$

Let $\frac{k}{2^n} < x < \frac{k+1}{2^n}$, then

$$\begin{aligned} f_n(x) - f_{n-1}(x) = 2^n \Bigg[&\left(x - \frac{k}{2^n}\right)\left[f_n\left(\frac{k+1}{2^n}\right) - f_{n-1}\left(\frac{k+1}{2^n}\right)\right] \\ &+ \left(\frac{k+1}{2^n} - x\right)\left[f_n\left(\frac{k}{2^n}\right) - f_{n-1}\left(\frac{k+1}{2^n}\right)\right]\Bigg] \end{aligned}$$

i.e $f_n(x) - f_{n-1}(x)$ is a weighted average of

$$f_n\left(\frac{k}{2^n}\right) - f_{n-1}\left(\frac{k}{2^n}\right) \text{ and } f_n\left(\frac{k+1}{2^n}\right) - f_{n-1}\left(\frac{k+1}{2^n}\right)$$

So the sup is attend on a point of the type of $\frac{k}{2^n}$.
If k is even then $f_n(\frac{k}{2^n}) = f_{n-1}(\frac{k}{2^n})$

$$\Rightarrow \sup_{x\in[0,1]} |f_n(x) - f_{n-1}(x)| = \max_{k \text{ odd } 0<k<2^n} \left| f_n\left(\frac{k}{2^n}\right) - f_{n-1}\left(\frac{k}{2^n}\right)\right|$$

So

$$\begin{aligned} \mathrm{P}\left(\sup_{x\in[0,1]} |f_n(x) - f_{n-1}(x)| > \frac{1}{n^2}\right) &= \mathrm{P}\left(\max_{k \text{ odd}} \left| f_n\left(\frac{k}{2^n}\right) - f_{n-1}\left(\frac{k}{2^n}\right)\right| > \frac{1}{n^2}\right) \\ &\leq \sum_{k \text{ odd } 0<k<2^n} \mathrm{P}\left(\left| f_n\left(\frac{k}{2^n}\right) - f_{n-1}\left(\frac{k}{2^n}\right)\right| > \frac{1}{n^2}\right) \\ &= \sum_{k \text{ odd } 0<k<2^n} \mathrm{P}\left(|Z_{\frac{k}{2^n}}| > \frac{1}{n^2 C_n}\right) \\ &= 2^{n-1} \cdot \mathrm{P}\left(|\mathcal{N}(0,1)| > \frac{\sqrt{2^{n+1}}}{n^2}\right) \\ &\leq 2^{n-1} \cdot C \cdot \exp\left(-\frac{2^{n+1}}{n^4}\right) \end{aligned}$$

$$\Rightarrow \sum_{n=1}^{\infty} \mathrm{P}\left(d(f_n, f_{n-1})) > \frac{1}{n^2}\right) < \infty$$

$$\underbrace{\Rightarrow}_{\text{Borel Cantelli}} \text{ w.p. } 1,\ \exists\, N\ s.t\, \forall\, n > N\ \ d(f_n, f_{n-1}) < \frac{1}{n^2}$$

$$\Rightarrow \text{ w.p } 1,\ (f_n) \text{ satisfies Cauchy's condtition}$$

$$\Rightarrow \text{ converges a.s. uniformly}$$

■

We constructed BM but did not prove that our object is indeed BM.

Reminder:
We took

$$C_n = \sqrt{\frac{1}{2^m}} \text{ for every dyadic number } y \in [0,1].$$

We also had a variable

$$Z_y \sim \mathcal{N}(0,1),\ (Z_y) \text{ iid.}$$

We defined f_0 as follows: $f_0(0) = 0$ $f_0(i) = Z$, in between linear interpolation. Recursiely we defined f_n ,

$$\begin{aligned} f_n\left(\tfrac{k}{2^n}\right) &:= f_{n-1}\left(\tfrac{k}{2^n}\right) && \text{for } k \text{ even} \\ f_n\left(\tfrac{k}{2^n}\right) &:= f_{n-1}\left(\tfrac{k}{2^n}\right) + c_n Z_{\frac{k}{2^n}} && \text{for } k \text{ odd} \end{aligned}$$

and in between we linearly interpolate. We proved that w.p.I the sequence f_n converges uniformly and we named the limit f. Obviously f is continious (it is the uniform limit of continious functions).

3.4 Theorem

We need to prove that f is BM (i.e that the diet of f is a Wiener measure) i.e

1. $P(f(0) = 0) = 1$
2. *for* $t > s$ $f(t) - f(s) \sim \mathcal{N}(0, t-s)$
3. *if* $t_1 < t_2 < ... < t_k$ *then* $\left(\delta(t_{l+1}) - \delta(t_l)\right)_{l=1}^{k-1}$ *are independent*

Proof:

1. trivial

Before we do (2) and (3) we need some discussion. Note that for every n and $0 \le k \le 2^n$, $f(\frac{k}{2^n}) = f_n(\frac{k}{2^n})$. Why?

$$f_{n+1}\left(\frac{k}{2^n}\right) = f_{n+1}\left(\frac{2k}{2^{n+1}}\right) = f_n\left(\frac{2k}{2^{n+1}}\right) = f_n\left(\frac{k}{2^n}\right)$$

similarly,

$$f_n\left(\frac{k}{2^n}\right) = f_{n+1}\left(\frac{k}{2^n}\right) = f_{n+2}\left(\frac{k}{2^n}\right) = f_{n+3}\left(\frac{k}{2^n}\right) = ... \Rightarrow f\left(\frac{k}{2^n}\right) = f_n\left(\frac{k}{2^n}\right)$$

3.5 Claim

$\Big(f_n(\frac{k+1}{2^n}) - f_n(\frac{k}{2^n})\Big)_{k=0}^{2^{n-1}}$ *are i.i.d* $\mathcal{N}(0, \frac{1}{2^n})$.

Proof:

We do this by introduction on n. Base case $n = 0$: we need to show $f_0(1) - f_0(0) \sim \mathcal{N}(0, 1)$. This is true. Now assume that this holds for $n-1$. For easement of notation write $X_{n,k} = f_n\Big(\frac{k+1}{2^n}\Big) - f_n\Big(\frac{k}{2^n}\Big)$. By introduction hypothesis,

$$X_{n-1,0}, X_{n-1,1}, ..., X_{n-1,2^{n-1}-1}$$

are all iid $\mathcal{N}(0, \frac{1}{2^{n-1}})$. Take $k = 0, ..., 2^{n-1}-1$. What are $X_{n,2k}$ and $X_{n,2k+1}$?

$$\begin{aligned}
X_{n,2k} &= f_n\left(\frac{2k+1}{2^n}\right) - f_n\left(\frac{2k}{2^n}\right) \\
&= f_{n-1}\left(\frac{2k+1}{2^n}\right) + c_n Z_{\frac{2k+1}{2^n}} - f_{n+1}\left(\frac{2k}{2^n}\right) \\
&= \frac{1}{2}\left[f_{n-1}\left(\frac{2k}{2^n}\right) + f_{n-1}\left(\frac{2k+2}{2^n}\right)\right] + c_n Z_{\frac{2k+1}{2^n}} - f_{n-1}\left(\frac{2k}{2^n}\right) \\
&= \frac{1}{2}\left[f_{n-1}\left(\frac{2k+2}{2^n}\right) - f_{n-1}\left(\frac{2k}{2^n}\right)\right] + c_n Z_{\frac{2k+1}{2^n}} \\
&= \frac{1}{2}\left[f_{n-1}\left(\frac{k+1}{2^{n-1}}\right) - f_{n-1}\left(\frac{k}{2^{n-1}}\right)\right] + c_n Z_{\frac{2k+1}{2^n}} \\
&= \frac{1}{2} X_{n-1,k} + c_n Z_{\frac{2k+1}{2^n}}
\end{aligned}$$

Similarly $X_{n,2k+1} = \frac{1}{2} X_{n+1,k} - c_n Z_{\frac{2k+1}{2^n}}$.

First Conclusion:

The pairs $(X_{n,2k}, X_{n,2k+1})$ are independent of each other. Why is

$$\underbrace{(X_{n,0}, X_{n,1})}_{\text{determined by } X_{n+1,0} \text{ and } Z_{\frac{1}{2^n}}}$$

independent to

$$\underbrace{(X_{n,2}, X_{n,3})}_{\text{determined by } X_{n-1,1} \text{ and } Z_{\frac{3}{2^n}}} \qquad ?$$

by introduction hypothesis all those variables are independent $\Rightarrow$ pairs are independent so we need to see that for given k, $X_{n,2k}$ and $X_{n,2k+1}$ are independent and both distributed according to $\mathcal{N}(0, \frac{1}{2^n})$.

Let

$$Y_i = X_{n-1,k} \cdot \sqrt{2^{n-1}}\ ,\ Y_2 = Z_{\frac{2k+1}{2^n}} \quad Y_1, Y_2 \text{ are iid } \mathcal{N}(0,1)$$

let

$$W_1 = \frac{Y_1 + Y_2}{\sqrt{2}}\ ,\ W_2 = \frac{Y_1 - Y_2}{\sqrt{2}} \text{ then } W_1, W_2 \text{ are independent } \mathcal{N}(0,1).$$

$$X_{n,2k} = \frac{1}{2}X_{n-1,k} + \frac{1}{\sqrt{2^{n-1}}} \cdot Z_{2^{k+1}} = \frac{1}{\sqrt{2^{n-1}}}Y_1 + \frac{1}{\sqrt{2^{n-1}}}Y_2 = \frac{1}{\sqrt{2^n}}W_1$$

Similarly

$$X_{n,2k+1} = \frac{1}{\sqrt{2^n}}W_2$$

so $X_{n,2k}$ and $X_{n,2k+1}$ are independent $\mathcal{N}(0, \frac{1}{2^n})$ and the claim is proved. ■

Now we can prove that f satisfies properties (2) and (3).

2. We need to show that for every

$$t > s \; f(t) - f(s) \sim \mathcal{N}(0, t - s)$$

If $k > j$

$$f(\frac{k}{2^n}) - f(\frac{j}{2^n}) = \sum_{k=j}^{k-1} X_{n,k} \sim \mathcal{N}(0, \frac{k}{2^n} - \frac{j}{2^n}$$

Let $t > s$ and let $(k_l), (j_l), (n_l)$ be seq. $s.t.$ $\frac{k_l}{2^{n_l}} \to t$ and $\frac{j_l}{2^{n_l}} \to S$ by cont. of f.

3. Is done exactly the same (End of prove). ■

3.6 Theorem

Wiener measure is unique.

Proof:
Of fact: $Cov(\delta(t), f(s)) = \min(s,t)$ Why? Assume $t > s$

$$Cov(f(t), f(s)) = Cov(f(s), f(s)) + Cov(f(t) - f(s), f(s)) = Var(f(s)) - 0 = s$$

but furthermore $\Rightarrow$ for every $t_1 < t_2 < ...t_k$ distribution of $f(t_l), ..., f(t_k)$ is MVG w. a known covariance Matrix

$$\underbrace{\Rightarrow}_{\text{Covariance Matrix determines distrib.}}$$

for all wiener measure. $(f(t_1), ..., f(t_k))$ have same joint distribution $\Rightarrow$ dist. $(f(t)$: dyadic$)$ is same for all wiener measures by continuity, $(\delta(t)$: t is dyadic$)$ determines. ■

3.7 Definition (Def of BM on $[0, \infty)$)

Let $\Omega := C[0, \infty)$ *(space of continuous functions* $[0, \infty) \to \mathbb{R}$*), with the topology of uniform convergence on every bounded interval.*

I.e. we say a sequence $f_n \to f$ is for every M, $f_n \to f$ on the interval $[0, M]$

$$f_n(x) = \frac{x}{n} \;, \; f_n \not\to f$$

however, for every M, $f_n \to f$ on $[0, M]$, so $\delta \to f$ in our topology.

3.8 Definition

Let $\mathcal{B}$ be borel σ-algebra. A measure P on $(\Omega, \mathcal{B})$ is a Wiener measure if

1. $P(f(0)) = 1$
2. *for* $0 \le t_1, ..., < t_n$

$$f(t_2) - f(t_1), f(t_3) - f(t_2), ..., f(t_n) - f(t_{n-1})$$

   are independent.

3. *for* $0 \le s < t,\ f(t) - \delta(s) \sim \mathcal{N}(0, t-s)$

We need to prove existence and uniqueness. Uniqueness is exactly same proof as for $[0, 1]$.

Proof of Existence:
Let $f^{(0)}, f^{(1)}, f^{(2)}, ..., : [0, 1] \to \mathbb{R}$ be independent BM. Now define for every

$$x \in [0, \infty)\ \ n(x) := [x]\ ,\ y(x) := x - [x]$$

then

$$f(x) := f_{n[x]}(y(x)) + \sum_{k=0}^{n[x]-1} f_k(i)$$

If

$$0 \le x < 1 \Rightarrow f(x) = f^{(0)}(x)$$

If

$$1 \le x < 2 \Rightarrow f(x) = f^{(1)}(x-1) + f^{(0)}(1)$$

and so it goes further. ■

Exercise: Check that it is a Wiener measure.

Notation: BM will be denoted $B(t)$ or B_t.

3.9 Theorem

With probability 1, B_t is nowhere differentiable. We prove that with prob. 1, B is non-differentiable for every $0 < t < 1$. From here the theorem follows once we note that B on $[0, \infty)$ is composed of independent copies of BM on $[0, 1]$.

Assume that B is differentiable at a point $t \in (0, 1)$. Then there exists integer M s.t. for every $y \in [0, 1]$

$$\left| \frac{B(y) - B(t)}{y - t} \right| < M.$$

Why? Let $B(t) = k$, then $\exists\, \epsilon > 0$ s.t if

$$|y - t| < \epsilon \text{ then } \left| \frac{B(y) - B(t)}{y - t} \right| < |k| + 10$$

Furthermore, $\exists\, K$ s.t $|B(y)| < K$ for every $y \in [0,1]$. So if $|y-t| \geq \epsilon$ then

$$\left|\frac{B(y)-B(t)}{y-t}\right| \leq \frac{2k}{\epsilon}$$

so if we take

$$M > \max\left(|k|+10, \frac{2k}{\epsilon}\right)$$

we are done. Therefore, if we show that for every integer M,

$$\mathrm{P}\left(\exists\, t \in (0,1):\ \forall y \neq t \in [0,1]\ \left|\frac{B(y)-B(t)}{y-t}\right| \leq M\right) = 0$$

we are done. Fix M. Call the event above A_M. We want to show $\mathrm{P}(A_M) = 0$.

Fix n, and consider the intervals

$$I_{n,k} = \left[\frac{k}{n}, \frac{k+1}{n}\right]$$

for $k = 0, ..., n-1$. If $t \in I_{n,k}$ $(k \leq n-3)$ and satisfies the property that $\forall\, y \neq t\ [0,1]$

$$\left|\frac{B(y)-B(t)}{y-t}\right| < M$$

Then for every

$$y \in I_{n,k}\ |B(y)-B(t)| \leq M|y-t| \leq M \cdot \frac{1}{n}$$

If $y \in I_{n,k+1}$, then

$$|B(y)-B(t)| \leq M|y-t| \leq \frac{2M}{n}$$

If $y \in I_{n,k+2}$, then

$$|B(y)-B(t)| \leq M|y-t| \leq \frac{3M}{n}$$

So

$$\left|B\left(\frac{k+1}{n}\right) - B\left(\frac{k}{n}\right)\right| \leq \left|B\left(\frac{k+1}{n}\right) - B(t)\right| + \left|B\left(\frac{k}{n}\right) - B(t)\right| \leq \frac{2M}{n} \leq \frac{6M}{n}$$

$$\left|B\left(\frac{k+2}{n}\right) - B\left(\frac{k+1}{n}\right)\right| \leq \left|B\left(\frac{k+2}{n}\right) - B(t)\right| + \left|B\left(\frac{k+1}{n}\right) - B(t)\right| \leq \frac{4M}{n} \leq \frac{6M}{n}$$

$$\left|B\left(\frac{k+3}{n}\right) - B\left(\frac{k+2}{n}\right)\right| \leq ... \leq \frac{6M}{n}$$

$B\left(\frac{k+1}{n}\right) - B\left(\frac{k}{n}\right), B\left(\frac{k+2}{n}\right) - B\left(\frac{k+1}{n}\right), B\left(\frac{k+3}{n}\right) - B\left(\frac{k+2}{n}\right)$ are iid $\mathcal{N}\left(0, \frac{1}{n}\right)$

Let $A_{n,k}$ be the event $\exists\, t \in [\frac{k}{n}, \frac{k+1}{n}]$ s.t $\forall\, y \neq t$

$$\left|\frac{B(y)-B(t)}{y-t}\right| \leq M$$

$$\mathrm{P}(A_{n,k}) \le \mathrm{P}\left(B(\frac{k+1}{n}) - B(\frac{k}{n}) \le \frac{6M}{n};\ \left|B(\frac{k+2}{n}) - B(\frac{k+1}{n})\right| \le \frac{6M}{n};\right.$$

$$\left.\left|B(\frac{k+3}{n}) - B(\frac{k+2}{n})\right| \le \frac{6M}{n}\right)$$

$$= \mathrm{P}\left(\left|B(\frac{k+1}{n}) - B(\frac{k}{n})\right| \le \frac{6M}{n}\right)^3 \le \left(\frac{C_1(M)}{\sqrt(n)}\right)^3 = \frac{C_2(M)}{n^{\frac{3}{2}}}$$

Fix m and let $A_M^m = \{\exists\, t \in (0, 1 - \frac{1}{m})$ s.t ...$\}$ take n so large that

$$1 - \frac{1}{n} > 1 - \frac{1}{m}.$$

Now

$$A_M^m \subseteq \bigcup_{k=0}^{n-3} A_{n,k} \Rightarrow \mathrm{P}(A_M^m) \le \sum_{k=0}^{n-3} \mathrm{P}(A_{n,k}) \le \frac{C_2(M)}{n^{\frac{3}{2}}} \cdot n = \frac{C_2(M)}{\sqrt{n}}$$

$$\Rightarrow \mathrm{P}(A_M^m) = 0$$

$$A_M = \bigcup_{m=1}^{\infty} A_M^m \Rightarrow \mathrm{P}(A_m) \le \sum_{m=1}^{\infty} \mathrm{P}(A_M^m) = 0 \Rightarrow \mathrm{P}(A_M) = 0$$

■

4 Filtrations and martingales

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a prob. space, and let $(F_t)_{t\ge 0}$ be σ-algebras, s.t

1. $F_t \subseteq \mathcal{F}$
2. for $t > s$, $F_s \subseteq F_t$

4.1 Definition

*A **filtration** on a probability space $(\Omega, \mathcal{F}, P)$ is a family $(F(t))_{t\ge 0}$ σ-algebras s.t*

$$F(s) \subset F(t) \subset F\ s < t$$

4.2 Definition

A filtration $(F_t)_{t\ge 0}$ is continuous from the right at t_0 if

$$F_{t_0} = \bigcap_{t > t_0} F_t$$

It is continuous from the left at t_0 if

$$F_{t_0} = \sigma(F_t, t < t_0)$$

It is almost continuous from the right (or left) if the σ-algebras only differ by null sets.

4.3 Definition

Two stochastic processes $(X_t)_{t\geq 0}$ *and* $(Y_t)_{t\geq 0}$ *one said to be* ***independent*** *if* $\forall\ 0 \leq t_1 \leq ... \leq t_n$ *and* $0 \leq s_1 \leq ... \leq s_k$

$$(X_{t_1}, ..., X_{t_n}) \ \perp \ (Y_{s_1}, ..., Y_{s_k})$$

- $(\Omega, \mathcal{F}, \mathrm{P})$ + filtration is called **a filtered probability space**
- $(X_t)_{t>0}$ is **adapted** to $(F(t))_{t\geq 0}$ if X_t is measurable w.r.t $F(t)$

4.4 Theorem

Let $(B_t)_{t\geq 0}$ *be a Brownian Motion (BM) started in* $x \in \mathbb{R}$ *(or* $\mathbb{R}^d$*), then* $\forall s \geq 0$

$$(B_{t+s} - B_s)_{t\geq 0}$$

is BM started in 0 and independent of $(B_t)_{t\in[0,s]}$.

4.5 Definition

The ***natural filtration*** *associated to a BM* $(B_t)_{t\geq 0}$ *is*

$$F^{\circ}(t) = \sigma\left((B_s)_{s\in[0,t]}\right)$$

Interpretation: $F^{\circ}(t)$ contains all information of 'what happened' up to time t.

Problem: $(F^{\circ}(t))_{t\geq 0}$ is not right-continuous.

Exercise:
The RV

$$L = \overline{\lim}_{t\to 0} \frac{B(t)}{\sqrt{2t \log|\log(t)|}}$$

is measurable w.r.t

$$\bigcap_{s>0} F^{\circ}(s)$$

but is not w.r.t $F^{\circ}(0)$.

Solution:
Take the filtration $F^{+}(t) = \bigcap_{s>t} F^{\circ}(s)$.
Homework: Prove that $(F^{+}(t))_{t\geq 0}$ is right-cont.

Observation:
F° and F^{+} one 'almost the same': all events in $F^{+}(t)$ but not in $F^{0}(t)$ are trivial.
$(B_t)_{t\geq 0}$ is adapted to F° and F^{+}.

4.6 Theorem

$\forall s > 0$ *the process* $(B(t+s) - B(s))_{t\geq 0}$ *is independent of* $F^{*}(s)$.

Proof:
By continuity of BM:

$$B(t+s) - B(s) = \lim_{m\to\infty} B(t+s_m)_B(s_m) \;\; s_m \searrow s$$

Then $0 \leq t_1 < ... < t_m$

$$(B(t_1+s) - B(s_m), ..., B(t_m+s) - B(s))$$

$$= \underbrace{\lim_{m\to\infty} \underbrace{(B(t_1+s_m) - B(s_m), ..., B(t_m+s_m) - B(s_m))}_{\text{independent } F^+(s)}}_{\text{indep. } F^+(s)}$$

■

$(B(t+s))_{t\geq 0}$ conditioned on $F^+(s)$ is a BM started at point $B(s)$

4.7 Theorem (Blumenthal 0-1 law)

If $A \in F^+(0)$, then $P(A) \in \{0,1\}$.

Proof:
By Theorem 1.4.7, $A \in \sigma\left((B_t)_{t\geq 0}\right)$ is independent of $F^+(0)$. In particular, since $A \in F^+(0),\; A$ is independent of itself. Therefore $P(A) \in \{0,1\}$.
(Hint: $P(A) = P(A\cap A) = P(A)P(A) = P(A)^2$) ■

Homework:

$$\tau_1 = \inf\{t > 0 : \; B(t) > 0\}$$

$$\tau_2 = \inf\{t > 0 : \; B(t) = 0\}$$

Prove that P$(\tau_1 = 0) = 1 \;\; P(\tau_2 = 0) = 1$.

4.8 Proposition

$x \in \mathbb{R},\; A$ a tail event i.e.

$$A \in \tau = \bigcap_{t\geq 0} \sigma\left((B(s))_{s\geq t}\right)$$

Then $P_x(A) \in \{0,1\}$.

Proof:
(sketch) Blumenthal + time inversion.

4.9 Definition

A real-valued stochastic process $(X_t)_{t\geq 0}$ *is a martingale with respect to a filtration* $(F(t))_{t\geq 0}$ *if*

- $(X_t)_{t\geq 0}$ *is adopted to* $(F(t))_{t\geq 0}$
- $E(|X_t|) < \infty \;\; \forall\, t \geq 0$
- $\forall\, 0 \leq s \leq t$

$$\begin{aligned} &E(X_t|F(s)) = X_s && \textit{a.s.} && \\ &E(X_t|F(s)) \leq X_s && \textit{a.s.} && \textit{supermartingale} \\ &E(X_t|F(s)) \geq X_s && \textit{a.s.} && \textit{submartingale} \end{aligned}$$

Interpretation: (X_t) does not grow or decrease in mean.

5 Stopping Times

Is the Markov Property time if we restart the process at random times?

5.1 Definition

A random variable $T \in [0, \infty]$ *on a filtrated probability space is a* ***stopping time*** *w.r.t.* $(F(t))_{t\geq 0}$ *if*

$$\{T \leq t\} \in F(t) \qquad \forall\, t \geq 0$$

- Every stopping time w.r.t. $(F^\circ(t))_{t\geq 0}$ is a stopping time w.r.t. $(F^+(t))_{t\geq 0}$
- If $(F(t))_{t\geq 0}$ is right continuous it sufficies

  $$\{T < t\} \in F(t) \qquad t \geq 0$$

  Why? Homework.

Examples:

- Any deterministic time t is a stopping time.
- The first time a BM $(B_t)_{t\geq 0}$ hits some closed set C w.r.t. $(F^\circ(t))_{t\geq t}$. (f.e. $C = \{1\}$)
  Why?

  $$\{T \leq t\} = \underbrace{\bigcap_{m=1}^{\infty} \bigcup_{s\in\mathbb{Q}\cap(0,t)} \bigcup_{x\in\mathbb{Q}\cap C} \underbrace{\left\{B(s) \in \mathit{Ball}\left(x, \frac{1}{m}\right)\right\}}_{\in F^\circ(t)}}_{\in F^\circ(t)}$$

- The first time $(B(t))_{t\geq 0}$ hits an open set O is a stopping time w.r.t. $(F^+(t))_{t\geq 0}$ but <u>not</u> w.r.t. $(F^\circ(t))_{t\geq 0}$.

- First time t s.t.

$$\max_{s\in[0,t]} B(s) - \min_{s\in[0,t]} B(s) = 3$$

5.2 Theorem

For every a.s. finite stopping time T, the process $(B(T+t) - B(t))_{t\geq 0}$ is a BM independent of $F^+(T)$ where

$$F^+(t) = \{A \in F : A \cap \{T \leq t\} \in F^+(t)\}$$

Example:

- $T = \inf\{t \geq 0 : B(t) = 1\}$

5.3 Theorem (Strong Markov property)

Let T be an a.s finite stopping time for the natural filtration (F_t^+). Then the process

$$(B(T+t) - B(T))_t$$

is a Brownian Motion and is independent of

$$F_T^+ := \{A \subseteq \mathcal{F} \ : \forall\, t\ A \cap \{T \leq t\} \ \in \ F_t^+\}$$

Exercise: Play with this definition ;)

Example:

1. $$T := \inf_t B(t) = 1$$
$$A_1 = \{\forall\, t < T\ s.t\ B(t) = -1\}$$
$$A_2 = \{\text{for every } T < s < T+1,\ B(s) \neq 0\}$$

2. $$T = \inf_t |B(t)| = 1 \ \ A_1 = \{B(T) = 1\}$$

Proof of Strong Markov Property:

First assume T is a dyadic stopping time, i.e $\exists\, n$ s.t with prob. 1 $T = \frac{k}{2^n}$ for some k.

Let A_k be the event $\{T = \frac{k}{2^n}\}$. Then $\bigcup\limits_{k=0}^{\infty} A_k = \Omega$ (up to a null event).

A_k's are pairwise disjoint. Choose k

$$\left(B\left(\frac{k}{2^n} + t\right) - B\left(\frac{k}{2^n}\right)\right)_t$$

is a BM independent of $F^+_{\frac{k}{2^n}}$. For every

$$A \subseteq F_T^+, \quad \left(B\left(\frac{k}{2^n}+t\right) - B\left(\frac{k}{2^n}\right)\right)$$

is a BM independent of the event $A \cap A_k$ which is in $F^+_{\frac{k}{2^n}}$.

Therefore, for every k

$$\left(B\left(\frac{k}{2^n}+t\right) - B\left(\frac{k}{2^n}\right)\right)_t$$

is a BM independent of $A \cap A_k$

For every A, conditioned on $A_k \cap A$, $(B(T+t) - B(T))_t$ is a BM. Now, note that A is a joint union of the events $(A \cap A_k)_{k=1}^{\infty}$.

Because of that, conditioned on A, $(B(T+t) - B(t))_t$ is a BM, and since this holds for every $A \in F_T^+$ we get that $(B(T+t) - B(T))_t$ is BM independent of F_T^+.

Let T be a stopping time. For every n, define

$$T_n := 2^{-n} \lceil 2^n T \rceil$$

(i.e the closest n-dyadic number from above). T_n is a stopping time, $T_n \geq T$ and $T_n - T \leq 2^{-n}$.

$$F \subseteq F^+_{T_n}$$

(Homework exercise) $\Rightarrow$ for every n,

$$(B(T_n + t) - B(T_n))_t$$

is BM independent of F_T^+.

$$(B(T+t) - B(T))_t = \lim_{n\to\infty} (B(T_n+t) - B(T_n))_t$$

a.s convergence uniformly on every bounded interval.

$\Rightarrow$ $(B(T+t) - B(T))_t$ is BM independent of F_T^+. ■

Question: What is the distribution of $\max\limits_{0\leq t\leq 1} B(t)$?

More precisely, for given γ, what is P $\left(\max\limits_{0\leq t\leq 1} B(t) > \gamma\right)$?

Let $A_\gamma = \{\exists\ 0 \leq t \leq 1\ B(t) = \gamma\}$

$$\text{P}\left(B(1) > \gamma | A_\gamma\right) = \frac{1}{2} \qquad \text{P}\left(B(1) > \gamma | A_\gamma^c\right) = 0$$

$\Rightarrow$ P $(A_\gamma) = 2$P $(B(1) > \gamma)$

5.4 Theorem (Optional sampling theorem)

Let (M_t) be a continuous martingale, and let $S \leq T$ be stopping times, all w.r.t same filtration (F_t). Then

$$E\left(M_T | F_s\right) = M_s$$

if (at least) one of the following conditions is satisfied:

1. *T is bdd i.e* $\exists\, M$ *s.t* $P(T \leq N) = 1$
2. *T is a.s finite and* (M_t) *uniformly integrable.*
3. $E(T) < \infty$ *and* $\exists\, M$ *s.t* $\forall\, t$ $E\left(|M_{t+1} - M_t|\Big|F_t\right) < M$

Application: Reflection principle.

5.5 Theorem

Let $(B_t)_{t\geq 0}$ *be a BM. Then* $\forall\, y > 0$

$$P\left(\max_{0\leq t\leq 1} B(t) > y\right) = 2P(B_1 > y)$$

Proof:
We start with the following fact: Let $(W_t)_{t\geq 0}$ be a BM, and let S be a random variable s.t. S is independent of $(W_t)_{t\geq 0}$, and $P(0 < S < \infty) = 1$. Then $P(W_S > 0) = \frac{1}{2}$ (Exercise).

Now, let $T = \min\{t \;:\; B_t \geq y\}$, and let $A = \{T < 1\}$.

5.6 Claim

$$P(A) = P\left(\max_{0\leq t\leq 1} B_t \geq y\right)$$

On the event A, let $W_t := (B_{T+t} - B_T)$, and let $S = 1 - T$. Then $(W_t)_{t\geq 0}$ is a BM independent of F_T (strong Markov Property) and S is measurable w.r.t. F_T.

$\Rightarrow$ S is independent of $(W_t)_{t\geq 0}$, and therefore $P(W_S) = \frac{1}{2}$.

Now on the event A,

$$B_1 - y = B_1 - B_T = W_{1-T} = W_S$$

$\Rightarrow B_1 > y$ if and only if $W_S > 0$ Thus,

$$P\left(B_1 > y\Big|A\right) = P\left(W_S > 0\Big|A\right) = \frac{1}{2}$$
$$P\left(B_1 > y\Big|A^c\right) = 0$$
$$\Rightarrow P(B_1 > y) = \frac{1}{2}P(A) = \frac{1}{2}P\left(\max_{0\leq t\leq 1} B_t > y\right)$$

■

4 Lemmas from Discrete Probability:

5.7 Lemma

Let (X_n) *be a sequence of random variables and let* X *be a RV. Assume:*

(i) $X_n \xrightarrow{a.s} X$

(ii) (X_n) *is unit. integrable (UI)*

Then

$$X_n \xrightarrow{\text{in } L'} X$$

5.8 Lemma

Let $(F_n)_{n=1}^{\infty}$ *be* σ*-algebras. Let* X *a RV such that* $E(|X|) < \infty$ *and for every* n *let*

$$X_n := E(X|F_n)$$

Then $(X_n)_{n=1}^{\infty}$ *is UI.*

Proof:

Reminder: We say that $(X_n)_{n=1}^{\infty}$ is UI. If for every ϵ $\exists$ M s.t $\forall$ n:

$$E(|X_n| 1_{|X_n|>M}) < \epsilon$$

Let $L = E(|X|)$. Let $Y_n = E(|X| \Big| F_n)$. Note that $Y_n \geq |X_n|$ a.s (Jensen). Fix ϵ and let M be so large that

$$E(|X| \ 1_{|X|>M\epsilon}) < \epsilon$$

Now,

$$\begin{aligned} E\left(|X_n| 1_{|X_n|>M}\right) \leq E\left(Y_n 1_{Y_n>M}\right) &= E\left(|X| 1_{Y_n>M}\right) \\ &= E\left(X \cdot 1_{Y>M} \cdot 1_{|X|>\epsilon M}\right) + E\left(|X| \cdot 1_{Y_n>M} \cdot 1_{|X|\leq \epsilon M}\right) \\ &\leq E\left(|X| \cdot 1_{|X|>\epsilon M}\right) + \epsilon \cdot M \cdot P(Y_n > M) \\ &\leq \epsilon + \epsilon M \cdot \frac{E(Y_n)}{M} \\ &= \epsilon \cdot (1 + L) \end{aligned}$$

■

5.9 Lemma

Let $(X_n)_{n=1}^{\infty}$ *UI. Then* $\forall \epsilon$ $\exists$ δ *s.t* $\forall$ n *and every event* A *s.t*

$$P(A) < \delta \qquad E(|X_n| \cdot \mathbb{1}_A) < \epsilon$$

Proof:

Let $(X_n)_{n=1}^{\infty}$ be UI. Fix ϵ. Let M be s.t. $\forall$ n:

$$E\left(|X_n| \cdot \mathbb{1}_{|X_n|>M}\right) < \epsilon.$$

Take δ so small s.t. $\delta \cdot M < \epsilon$. Fix n. Let A be an event s.t. $P(A) < \delta$,and let $A_1 = A \cap \{|X_n| \leq M\}$ and $A_2 = A \cap \{|X_n| > M\}$. Now

$$\begin{aligned} E\left(|X_n| \cdot \mathbb{1}_A\right) &= E\left(|X_n| \cdot \mathbb{1}_A\right) + E\left(|X_n| \cdot \mathbb{1}_{A_2}\right) \\ &\leq P(A_1) \cdot M + E\left(|X_n| \cdot \mathbb{1}_{|X_n|>M}\right) \\ &\leq P(A) \cdot M + E\left(|X_n| \cdot \mathbb{1}_{|X_n|>M}\right) \\ &< \delta M + \epsilon < 2\epsilon \end{aligned}$$

■

5.10 Lemma

Let $(M(t))_{t=0}$ be a continuous martingale, and let let T be a stopping time, both w.r.t same Filtration (F_t). Let $(Y(t))_{t>0}$ be stopped martingale $(Y(t) = M(t \wedge T))_{t>0}$. Then $(Y(t))$ is a martingale.

Proof:

For every $n = 1, 2, ...$ let

$$T_n = 2^{-n} \lceil 2^n T \rceil$$

the n-th dyadic approximation of T. Then T_n is a stopping time, $T_n \geq T$ and $T_n - T \leq 2^n$. In particular, a.s $\lim_{n\to\infty} T_n = T$. Let $Y_n(t) = M(t \wedge T_n)$. We first show (as in Probability Theory course) that Y_n is a martingale. We do this in two steps:

(1) We show that for every $k = 0, 1...$

$$E\left(Y_n\left(\frac{k+1}{2^n}\right)\Big|F_{\frac{k}{2^n}}\right) = Y_n\left(\frac{k}{2^n}\right) \quad \text{a.s}$$

$$\begin{aligned}
E\left(Y_n\left(\frac{k+1}{2^n}\right)\Big|F_{\frac{k}{2^n}}\right) &= E\left(Y_n\left(\frac{k+1}{2^n}\right)\cdot \mathbb{1}_{T_n\geq\frac{k+1}{2^n}}\Big|F_{\frac{k}{2^n}}\right) + E\left(Y_n\left(\frac{k+1}{2^n}\right)\cdot \mathbb{1}_{T_n\leq\frac{k}{2^n}}\Big|F_{\frac{k}{2^n}}\right)\\
&= E\left(M\left(\frac{k+1}{2^n}\right)\cdot \mathbb{1}_{T_n\geq\frac{k+1}{2^n}}\Big|F_{\frac{k}{2^n}}\right) + E\left(M_{T_n}\cdot \mathbb{1}_{T_n\leq\frac{k}{2^n}}\Big|F_{\frac{k}{2^n}}\right)\\
&= M\left(\frac{k}{2^n}\right)\cdot \mathbb{1}_{T_n\geq\frac{k+1}{2^n}} + M(T_n)\cdot \mathbb{1}_{T_n\leq\frac{k}{2^n}}\\
&= Y_n\left(\frac{k}{2^n}\right)
\end{aligned}$$

$\Rightarrow$ for $k > j$

$$E\left(Y_n\left(\frac{k}{2^n}\right)\Big|F_{\frac{j}{2^n}}\right) = Y_n\left(\frac{j}{2^n}\right)$$

(2) Let $t > s$. We want to show

$$E\left(Y_n(t)\Big|F_s\right) = Y_n(s)$$

Exercise!

Now fix $s < t$ we want to prove

$$E\left(Y_{(t)}\Big|F_s\right) = Y_{(s)}$$

We need two statements:

(i) $Y_n(t) \xrightarrow{a.s} Y(t)$ and $Y_n(s) \xrightarrow{a.s} Y(s)$.

(ii) $(Y_n(t))_{n=1}^{\infty}$ is UI.

Assume that the statements hold. Then first $Y_n(t) \xrightarrow{\text{L'}} Y(t)$ and second $Y_n(s) \xrightarrow{\text{L' a.s}} Y(s)$.

$$\xrightarrow{\text{(i)}} E\left(Y(t)\middle|F_s\right) \overset{\text{L'}}{=} \lim_{n\to\infty} E\left(Y_n(t)\middle|\mathcal{F}_s\right) \overset{\text{L'}}{=} \lim_{n\to\infty} \left(Y_n(s)\right) = Y(s)$$

Proof of statements:

(i)

$$\begin{aligned} \lim_{n\to\infty} Y_n(t) &= \lim_{n\to\infty} M(t \wedge T_n) \\ &= M(\lim_{n\to\infty} t \wedge T_n) \\ &= M(t \wedge \lim_{n\to\infty} T_n) \\ &= M(t \wedge T) = Y(t) \end{aligned}$$

all limits are a.s limits, because $T_n \xrightarrow{\text{a.s}} T$.

(ii) We will show that

$$Y_n(t) = E\left(Y_1(t)\middle|\mathcal{F}_{t\wedge T_n}\right).$$

Then it will follow from Lemma 5.8.

$$\begin{aligned} E\left(Y_1(t)\middle|F_{T_n\wedge t}\right) &= E\left(M_{t\wedge T_1}\middle|F_{T_n\wedge t}\right) \\ &= \sum_{k=0}^{\infty} \mathbb{1}_{T_n=\frac{k}{2^n}} E\left(M_{t\wedge T_n}\middle|F_{T_n\wedge t}\right) \\ &= \sum_{k=0}^{\infty} \mathbb{1}_{T_n=\frac{k}{2^n};t\leq T_n} E\left(M_{t\wedge T_n}\middle|F_{T_n\wedge t}\right) + \mathbb{1}_{T_n=\frac{k}{2^n};t>T_n} E\left(M_{t\wedge T_n}\middle|F_{T_n\wedge t}\right) \\ &= \sum_{k=0}^{\infty} \mathbb{1}_{T_n=\frac{k}{2^n};t\leq T_n} E\left(M_t \cdot \mathbb{1}_{t\leq T_n}\middle|F_t\right) + \mathbb{1}_{T_n=\frac{k}{2^n}\ ;t>T_n} E\underbrace{\left(M_{t\wedge T} \cdot \mathbb{1}_{t>T_n}\middle|F_{\frac{k}{2^n}}\right)}_{M_{\frac{k}{2^n}}} \\ &= \sum_{k=0}^{\infty} \mathbb{1}_{T_n=\frac{k}{2^n};t\leq T_n} M_{t\wedge T_n} + \mathbb{1}_{T_n=\frac{k}{2^n};t>T_n} M_{T_n\wedge t} \\ &= Y_n(t) \end{aligned}$$

■

5.11 Lemma

Let $(M_t)_{t\geq 0}$ be a continuous Martingale and let S be a stopping time, both w.r.t (F_t). Assume further:

1. $P(S < \infty) = 1$
2. $\forall\, t\ ,\ E\left(|M_{s+t}|\right) < \infty$

Then $(Y_t := M_{s+t})$ is a Martingale w.r.t $(G_t = F_{t+s})_{t>0}$

Proof:

1. Y_t means w.r.t G_t- obvious.
2. $E(|Y_t|) < \infty$ from assumption
3. $E(|Y_t \Big| G_s) = Y_s$ for $s < t$. We will do it in two steps:
   (a) If S is dyadic (i.e $\exists\, n\ s.t.\ w.p.1\ S \cdot 2^n \in \mathbb{N}$).
   (b) From dyadic to general stopping times using approximation $S_n = 2^{-n}\lceil 2^n S\rceil$ (this we will skip).

   (a)

$$\begin{aligned} E\left(Y_t \Big| G_s\right) &= E\left(M_{S+s} \Big| F_{S+s}\right) \\ &= \sum_{k=0}^{\infty} \mathbb{1}_{S=\frac{k}{2^n}} E\left(M_{S+t} | F_{S+s}\right) \\ &= \sum_{k=0}^{\infty} \mathbb{1}_{S=\frac{k}{2^n}} E\left(M_{\frac{k}{2^n}+t} \Big| F_{\frac{k}{2^n}+t}\right) \\ &= \sum_{k=0}^{\infty} \mathbb{1}_{S=\frac{k}{2^n}} \cdot M_{\frac{k}{2^n}+s} \\ &= M_{S+s} \\ &= Y_s \end{aligned}$$

5.12 Theorem (Optional stopping theorem)

Let (M_t) be a continuous Martingale, and let $S < T$ be finite stopping times, all w.r.t (F_t). Then

$$E\left(M_T \Big| F_S\right) = M_S \ \ \textit{if} \ \ (M_{T\wedge L})_{L>0} \ \ \textit{is UI}$$

In particular this happens if one of the following conditions holds:

1. $\exists\, M\ s.t\ \ P(T < M) = 1$
2. $(M_t)_{t>0}$ is U.I
3. $E(T) < \infty$ and $\exists\, M$ s.t $\ \forall\, t\ E\left(|M_{t+1} - M_t| F_t\right) < M$

Sketch of proof of OST

1. Let $N_t = M_{s+t}$. We need to show (N_t) is a Martingale. To do this it suffices to show that $E\left(|M_{s+t}|\right) < \infty$, for every t. For this we follow the same method as before:
   (a) For S dyadic decompose according to the value of S. Show that it goes through as $S_n \to S\ ,\ S_n = 2^{-n}\lceil 2^n S\rceil$, by UI.

(b) Let $T = T - S$. Then $M_T = N_\tau$, τ is stopping time and $(N_{T\wedge L})_{L>0}$ is UI (exercise: verify). Now, let

$$Y^{(L)}(t) = N_{t\wedge\tau\wedge L}$$

$Y^{(L)}(t)$ is a Martingale, $Y^{(L)}(L) = N_{\tau\wedge L}$ so

$$E\left(Y^{(L)}\Big|G_o\right) = Y^{(L)}(0) = Y(0) = M_S.$$

Note that $Y^{(L)}(\tau) \underset{\text{a.s}}{\overrightarrow{\frown}} N_\tau$ and due to UI also in L'.

$$\Rightarrow\ E\left(N_\tau\Big|F-0\right) = \lim_{n\to\infty} E\left(Y^{(L)}(\tau)|G_o\right) = \lim_{L\to\infty} E\left(Y^{(L)}(L)\Big|G_o\right) = N_o = M_s$$

Now we need to show that each of the conditions (1)-(3) yields UI of $(M_{T\wedge L})_{L>0}$. We postpone this proof.

6 Quadratic variation

6.1 Definition

*Let $[a,b]$ be an interval. A **partition** Π of $[a,b]$ is a finite set*

$$\Pi = \{t_0 = a < t_1 < t_2 < ... < t_n = b\}$$

We also need the operators

$$len(\Pi) = |\Pi| - 1 \qquad mesh(\Pi) = \max_{k=0,\dots,len(\Pi)-1} t_{k+1} - t_k$$

A partition Π is a **refinement** of M_2 if $M_1 \geq M_2$.

A **refining sequence of partitions** is a sequence of partitions

(i) $(\Pi_n)_{n=1}^\infty$ s.t. $\Pi_{n+1} \geq \Pi_n\ \forall\ n$

(ii) $\lim_{n\to\infty} mesh(\Pi_n) = 0$

6.2 Definition

*$f : (0,t) \to \mathbb{R}$ is of **bounded variation** if*

$$\sup_{\Pi=\{t_0,\dots,t_n\}} \sum_{k=0}^{len(\Pi)} |f(t_{k+1}) - f(t_k)| < \infty$$

6.3 Definition

*Let $(M_t)_{t>0}$ be a stochastic process and let (X_t) be an other process. We say that (X_t) is the **quadratic variation** of (M_t) if for every t and for every refining sequence of partitions (Π_n) of $[0,t]$ we have*

$$\sum_{k=0}^{len(\Pi)} \left(M_{t_{k+1}} - M_{t_k}\right)^2 \xrightarrow{n\to\infty} X_t \qquad a.s.$$

6.4 Fact

Let (M_t) be deterministic, continuous with bounded variation. The quadratic variation of (M_t) is zero.

Proof:
(M_t) is continuous, therefore it is uniformly continuous because we are on a compact interval. Because of that there exists for every ε a δ s.t. if $mesh(\Pi_n) < \delta$ then for all ε it holds

$$\left|M_{t_{k+1}} - M_{t_k}\right| < \varepsilon$$

For such n

$$\begin{aligned}\sum_{k=0}^{len(\Pi_n)} \left(M_{t_{k+1}} - M_{t_k}\right)^2 &= \sum_{k=0}^{len(\Pi_n)} \left|M_{t_{k+1}} - M_{t_k}\right| \left|M_{t_{k+1}} - M_{t_k}\right| \\ &\leq \varepsilon \sum_{k=0}^{len(\Pi_n)} \left|M_{t_{k+1}} - M_{t_k}\right| \\ &\leq \varepsilon \cdot variation(M_t) \xrightarrow{n\to\infty} 0\end{aligned}$$

■

<u>Exercise:</u>
Prove directly that the BM has infinite variation.

6.5 Theorem

The quadratic variation of BM is $(X_t = t)$.

Proof:
We will show that the QV of BM in $[0,1]$ is 1. Generalization to general $t \to$ exercise.

Let (Π_n) be a refining sequence of partitions of $[0,1]$. Assume w.l.o.g. (without loss of generality) that $len(\Pi_n) = n$. Let

$$X_n = \sum_{k=0}^{len(\Pi_n)-1} (B(t_{k+1}) - B(t_k))^2$$

We need to prove $X_n \to 1$ a.s.. We do this in two steps:

1. Prove that (X_n) converges a.s..
2. Identify the limit and verify that it is 1.

1. Notation $\Pi_n = \{0 = t_0^n < T_1^n < \cdots < t_n^n = 1\}$. Let

$$G_n := \sigma\left(\left|B_{t_{k+1}^m} - B_{t_k^m}\right| : m \geq n, 0 \leq k \leq m-1\right)$$

   G_n is an inverse filtration i.e. $G_n \geq G_{n+1}\ \forall\ n$. We need to show:

   (i) $E(|X_n|) < \infty$

   (ii) X_n is measurable w.r.t. G_n.

(iii) $E(X_n|G_{n+1}) = X_{n+1}$

So

(i)

$$E(X_n) = \sum_{k=0}^{n-1} E\left[(B(t_{k+1}) - B(t_k))^2\right]$$
$$= \sum_{k=0}^{n-1} t_{k+1} - t_k = 1$$

(ii) This fact is obvious.

(iii) Remember Π_{n+1} was obtained by adding exactly one point to the partition Π_n. Let k be s.t. $t_k^n = t_k^{n+1}$ but $t_{k+1}^n > t_{k+1}^{n+1}$.

$$X_n = \left(B_{t_1^n} - B_{t_0^n}\right)^2 + \left(B_{t_2^n} - B_{t_1^n}\right)^2 + \cdots + \left(B_{t_{k+1}^n} - B_{t_k^n}\right)^2 + \cdots + (\dots)^2$$
$$X_{n+1} = \left(B_{t_1^n} - B_{t_0^n}\right)^2 + \left(B_{t_2^n} - B_{t_1^n}\right)^2 + \cdots + \left(B_{t_{k+1}^{n+1}} - B_{t_k^n}\right)^2 + \left(B_{t_{k+1}^n} - B_{t_{k+1}^{n+1}}\right)^2 + \cdots + (\dots)^2$$

Thus, all we need to show is that

$$E\left[\left(\underbrace{B_{t_{k+1}^n} - B_{t_k^n}}_{:=\alpha+\beta}\right)^2 \middle| G_{n+1}\right] = \left(\underbrace{B_{t_{k+1}^{n+1}} - B_{t_k^n}}_{:=\alpha}\right)^2 + \left(\underbrace{B_{t_{k+1}^n} - B_{t_{k+1}^{n+1}}}_{:=\beta}\right)^2$$

$|\alpha|$ is G_{n+1}-measurable, $|\beta|$ is G_{n+1}-measurable. However, conditioned on G_{n+1} with probability $\frac{1}{2}$ $\alpha \cdot \beta > 0$ and with probability $\frac{1}{2}$ $\alpha \cdot \beta < 0$. Furthermore with probability $\frac{1}{2}$ $\alpha \cdot \beta = |\alpha| \cdot |\beta|$ and with probability $\frac{1}{2}$ $\alpha \cdot \beta = -|\alpha| \cdot |\beta|$. Now

$$E\left((\alpha+\beta)^2|G_{n+1}\right) = E\left(\alpha^2 + \beta^2 + 2\alpha\beta|G_{n+1}\right)$$
$$= \alpha^2 + \beta^2 + 2E\left(\alpha\beta|G_{n+1}\right)$$
$$= \alpha^2 + \beta^2 + 2\left(\frac{1}{2}|\alpha||\beta| + \frac{1}{2}(-|\alpha||\beta|)\right)$$
$$= \alpha^2 + \beta^2$$

So X_n is an inverse martingale, so (X_n) converges a.s..

2. As exercise: $Var(X_n) \xrightarrow{n\to\infty} 0 \quad \Rightarrow \; X_n \xrightarrow{\text{in probability}} 1$ as it converges in probability to 1 and a.s. to something, the a.s. limit is 1.

■

7 Ito's Integral w.r.t Döblin

7.1 Definition

A function $f: \Omega \times [0,\infty) \to \mathbb{R}$ is called ***progressively measurable*** *if*

1. *f is measurable w.r.t $F \times B$, F being the standard filtration of Ω, B Borel-σ-algebra of $[0,\infty)$, and*
2. *$\sigma(f(s): 0 \le s \le t) \subset F_t = \sigma(B(s): 0 \le s \le t)$*

7.2 Definition

A random function $(f(t))_{t>0}$ is called a ***progressively measurable step function*** *(PMSF) if:*

(i) For every t $\sigma(f(s), s \le t) \subset F_t$ (F_t natural filtration of a BM).

(ii) There exist random numbers $0 < t_1 < \cdots < t_n$ s.t. f is constant on $[0,t_1)$, $[t_1,t_2)$, $\ldots$, $[t_{n-1},t_n)$ and f is zero on $[t_n,\infty)$.

(iii)

$$E\left(\int_0^\infty f^2(t)\,dt\right) < \infty$$

7.3 Definition

The Ito-integral for progressively measurable step function is defined as

$$\int_0^\infty f\,dB = \sum_{k=0}^{n-1} f(t_k)\left(B_{t_{k+1}} - B_{t_k}\right)$$

Let f be a PMSF. We write

$$\int_0^\infty f\,dB_t := \sum_{k=1}^{n} f(t_{k-1})\left[B(t_k) - B(t_{k-1})\right]$$

Exercise: Show that the integral is determined by the function (same value for different representations of the sequence).

7.4 Fact

1. *The space of PMSF is a linear subspace (not closed of V; V is a Hilbert space).*
2. *Let $I(f) := \int_0^\infty f dB_t$, then I is a linear transformation.*
3. *I is an isometry from PMSF to $L^2(\Omega)$ i.e*

$$E\left(I(f)^2\right) = E\left(\int_0^\infty f^2(t)dt\right)$$

4. *The space of PMSF is dense in V.*

Proof of facts:

1. obvious
2. $$I\left(\alpha \cdot f\right) = \sum_{k=1}^{\infty} \alpha \cdot f(t_{k-1})\left[B(t_k) - B(t_{k-1}\right] = \alpha \sum$$

   if f_1 and f_2 are PMSF, we can find $0 = t_0 < t_1 ... < t_n$ which represent f_1, f_2 and $f_1 + f_2$, and then

   $$\begin{aligned} I(f_1 + f_2) &= \sum_{k=1}^{n} \left[f_1(t_{k-1}) + f_2(t_{k-1})\right]\left[B(t_k) - B(t_{k-1})\right] \\ &= \sum f_1(t_{k-1})[...] + \sum f_2(t_{k-1})\left[B_B\right] \\ &= I(f_1) + I(f_2) \end{aligned}$$

3. Let f be a PMSF w.r.t $0 = t_0 < t_1 < ... < t_n$. $f(t_{k-1})$ is measurable w.r.t $F_{t_{k-1}} \Rightarrow \ f(t_k)$ is independent of $B(t_k) - B(t_{k-1})$

   $$\begin{aligned} \Rightarrow E\left(f^2(t_{k-1})\right) \cdot \left[(B(t_k) - B(t_{k-1}))^2\right] &= E\left(f^2(t_{k-1})\right) E\left[(B(t_k) - B(t_{k-1}))^2\right] \\ &= (t_k - t_{k-1})\, E\left(f^2(t_{k-1})\right). \end{aligned}$$

   let $j > k$

   $$\begin{aligned} E\left(\underbrace{f(t_{k-1})\left[B(t_k) - B(t_{k-1})\right] \cdot f(t_{j-1})}_{\text{measurable w.r.t} F_{t_{j-1}}} \cdot \left[B(t_j) - B(t_{j-1})\right]\right) &= E\left(B(t_j) - B(t_{j-1})\right) \cdot E\left(...\right) \\ &= 0 \end{aligned}$$

   Write $b_k = B(t_k) - B(t_{k-1})$

   $$\begin{aligned} E\left(I(f)^2\right) &= E\left[\left(\sum_{k=1}^{n} f(t_{k-1}) \cdot b_k\right)^2\right] \\ &= E\left[\sum_{k,j=1}^{n} f(t_{k+1}) f(t_{j-1}) b_k b_j\right] \\ &= \sum_{k=1}^{n} E\left(f^2(t_{k-1}) b_k^2\right) + 2\sum_{j>k} E\left(f(t_{k-1}) b_k f(t_{j-1}) b_j\right) \\ &= \sum_{k=1}^{n} E\left(f^2(t_{k-1})\right)\left[t_k - t_{k-1}\right] \\ &= E\left(\sum_{k=1}^{n} f^2(t_{k-1})[t_k - t_{k-1}]\right) \\ &= E\left(\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f^2(t)dt + \int_{t_n}^{\infty} f^2(t)dt\right) \\ &= E\left(\int_0^{\infty} f^2(t)dt\right) \end{aligned}$$

4. Let $U = \{f \in V : \exists\, M\ s.t.\ f(t) = 0\, \forall t > M\}$.

7.5 Claim

U is dense V.

Proof:
Let $f \in V$

$$\lim_{M\to\infty} E\left(\int_0^\infty f^2(t)\mathbb{1}_{t>M}dt\right) = 0$$

$$f^{(M)} := f(t)\cdot \mathbb{1}_{t\le M}\ \in\ U \quad \left\|f - f^{(M)}\right\|_2^2 = E\left(\int_0^\infty f^2(t)\mathbb{1}_{t>M}dt\right) \to 0 \text{ for } M\to\infty$$

So every f in V can be approx. by f_M in U.

7.6 Claim

PMSF is dense in U.

Proof:
Let $f\ \in\ U$, $M < t$ $f(t) = 0$ for $t > M$ Take $f^{(n)}$ as follows: for every t take

$$a_n(t) = 2^{-n}\llcorner 2^n\cdot t\lrcorner\ ,\ b_n(t) = 2^{-n}\ulcorner 2^n\cdot t\urcorner\ , f^{(n)}(t) = 2^n\int_{a_n(t)}^{b_n(t)} f(s)ds$$

$f^{(n)}$ is PMSF $\forall\, n$, $\left(f^{(n)}\right)$ is an L^2 martingale converging to f .

Now we define $I(f)$ for $f \in V$. $\exists\,(f_n)_{n=1}^\infty$ of PMSF converging in L^2 to f.

(f_n) converges $\to$ (f_n) is a cauchy sequence

$$\to \forall\,\epsilon\,\exists\, N\ s.t.\ \forall\, m > N\ ,\ E\left(\int_0^\infty (f_m - f_N)^2dt\right) < \varepsilon$$

$$\underbrace{\to}_{\text{I is isometry}}\ \forall\,\varepsilon\,\exists\, N\,\forall\, m > N\ E\left[(I(f_m) - I(f_N))^2\right] < \varepsilon$$

$$\to\ I(f_n) \underbrace{\Rightarrow}_{\text{is a cauchy sequence in} L(\Omega)} I(f_n) \text{ converges to some limit} X.$$

Define $\int_0^\infty f dB_t := X$.

■

Exercise: (i) Show that X does not depend on choice of sequence (ii) isometry

7.7 Theorem

Let $f \in V$. There exists a progressive measurable function F s.t.:

1. *F is a.s. continuous.*
2. *F is a martingale.*
3. *for every T, with probability 1*

$$F(T) = \int_0^\infty f(t)\cdot\mathbb{1}_{t<T}dB_t$$

4. *F is unique up to probability 0.*

8 Ito's formula

We start with Ito's formula for the independent one-dimensional systems.

8.1 Theorem

Let $g : \mathbb{R} \to \mathbb{R}$, and assume g is continuously twice differentiable and

$$E\left[\int_0^T [g'(B_s)]^2 ds\right] < \infty$$

and

$$P\left(\int_0^T |g''(B_s)| ds < \infty\right) = 1$$

for some $T > 0$. Then for every t,

$$g\left(B(t)\right) - g(0) = \int_0^t g'\left(B(s)\right) dB(s) + \frac{1}{2}\int_0^t g''\left(B(s)\right) ds$$

Note: if $f : \mathbb{R} \to \mathbb{R}$ has bounded variation, then

$$g(f(t)) - g(f(0)) = \int_0^t g'(f(s)) df(s) \text{ [standard chain rule]}$$

Proof of Ito's formula:

Analog assume $t = 1$. Fix n, and let

$$S_k^{(n)} = \frac{k}{2^n} \text{ while } h(s) = g\left(B(s)\right)$$

We want to understand $h(1) - h(0)$. We write

$$h(1) - h(0) = \sum_{k=0}^{2^n-1} h\left(S_{k+1}^{(n)}\right) - h\left(S_k^{(n)}\right)$$

$$\begin{aligned} h\left(S_{k+1}^{(n)}\right) - h\left(S_k^{(n)}\right) &= g\left(B(S_{k+1}^{(n)})\right) - g\left(B(S_k^{(n)})\right) \\ &= g'\left(S_k^{(n)}\right)\left[B\left(S_{k+1}^{(n)}\right) - B\left(S_k^{(n)}\right)\right] + \frac{1}{2}g''(\ldots)\left[B\left(S_{k+1}^{(n)}\right) - B\left(S_k^{(n)}\right)\right] \end{aligned}$$

for some $s \in \left[B\left(S_k^{(n)}\right), B\left(S_{k+1}^{(n)}\right)\right]$.

$$g\left(B(1)\right) - g(0) = \sum_{k=0}^{2^n-1} g'\left(B(S_k^{(n)})\right)\left[B\left(S_{k-1}^{(n)}\right) - B\left(S_k^{(n)}\right)\right] \quad (1)$$

$$+\frac{1}{2}\sum_{k=0}^{2^n-1} g''(s)\left[B\left(S_{k+1}^{(n)}\right) - B\left(S_k^{(n)}\right)\right]^2 \quad (2)$$

Now we need to show

1. $(1) \xrightarrow{n \to\infty} \int_0^1 g'\left(B(s)\right) dB(s)$
2. $(2) \xrightarrow{n\to\infty} \frac{1}{2}\int_0^1 g''\left(B(s)\right) ds$

1. Let $h^{(n)}(s) = g'\left(B(2^n \llcorner 2^n s \lrcorner)\right)$. By continuously of g' and of B,

$$\begin{aligned} h^{(n)}(s) &\to g'\left(B(s)\right) \\ &\Rightarrow \sum_{k=0}^{2^n-1} g'(B(S_k^{(n+1)} \left[B\left(S_{k+1}^{(n)}\right) B\left(S_k^{(n)}\right)\right] \\ &= \int_0^1 h^{(n)}(s) dB(s) \to_{n\to\infty} \int_0^1 g'\left(B(s)\right) dB(s) \end{aligned}$$

2. Remember that g'' is continuous in S. Thus for every ε exists N s.t. for every $0 \leq k \leq 2^{n-1}$ and every $S_k^{(N)} \leq S \leq S_{k+1}^{(N)}$, $|g(B(s)) - g''(B(S_k^{(N)}))| < \varepsilon$. Take $n \gg N$. For every $0 \leq k \leq 2^N$

$$\lim_{n\to\infty} \sum_{j=k2^{n-N}}^{(k+1)2^{n-N}-1} \left[B\left(S_{j+1}^{(n)}\right) - B\left(S_j^{(n)}\right)\right]^2 = 2^{-N}$$

$$\sum_{j=0}^{2^n-1} g''(B(s)) \left[B\left(S_j^{(n)}\right) - B\left(S_{j+1}^{(n)}\right)\right]^2 = \sum_{k=0}^{2^N-1} \left[\sum_{k2^{n-N}}^{(k+1)2^{n-N}-1} g''(B(s)) \left[B(S_{j+1}^{(k)}) - B\left(S_j^{(n)}\right)\right]\right]^2$$

For every k

$$\begin{aligned} \underbrace{\sum \left(g''\left(B\left(S_k^N\right)\right) - \varepsilon\right) [B(\cdot) - B(\cdot)]^2}_{(\star)} &\leq \sum_{k2^{n-N}}^{(k+1)2^{n-N}-1} g''(B(s)) \left[B\left(S_{j+1}^{(k)}\right) - B\left(S_j^{(n)}\right)\right]^2 \\ &\leq \underbrace{\sum \left(g''\left(B\left(S_k^N\right)\right) - \varepsilon\right) [B(\cdot) - B(\cdot)]^2}_{(\star\star)} \end{aligned}$$

So we compute $(\star)$ to

$$(\star) \xrightarrow{n\to\infty} g''\left(B\left(S_k^N\right) - \varepsilon\right) \cdot 2^{-N}$$

and

$$(\star\star) \xrightarrow{n\to\infty} g''\left(B\left(S_k^N\right) + \varepsilon\right) \cdot 2^{-N}$$

For every $S_k^{(N)} \leq S \leq S_{k+1}^{(N)}$

$$\left|g''(B(s)) - g''\left(B\left(S_k^{(N)}\right)\right)\right| < \varepsilon 2^{-N}$$

$\Rightarrow$ for all large enough n

$$\left|(2) - \int_0^1 g''(B(s))\, ds\right| < 2\varepsilon$$

so

$$(2) \xrightarrow{n\to\infty} \int_0^1 g''(B(s))\, ds$$

■

Exercise:
A bad student of Ito' said the following. In the Taylor expansion, say

$$g\left(B(S_{k+1}^{(n)})\right) - g\left(B(S_k^{(n)}\right) = g(s)\left[B\left(S_{k+1}^{(n)}\right) - B\left(S_k^{(n)}\right)\right]$$

follow calculations, and get

$$g\left(B(1)\right) - g\left(B(0)\right) = \int_0^1 g'\left(B(s)\right) dB(s)$$

Where exactly is the student wrong?

8.2 Definition

An n-dimensional BM is n independent BM's $B^{(1)}, ..., B^{(n)}$ The corresponding filtration is

$$F_t = \sigma\left(B^{(1)}(s), ...,^{(n)}(s) \ : 0 \le s \le t\right).$$

All theory is same. We can define

$$\int f \, dB^{(j)}(s)$$

for any $1 \le j \le n$.

Exercise: What is $\int_0^1 B^{(1)}(s)\, dB^{(2)}(s)$?

8.3 Definition

Multi dimensional Ito's formula: Let $g : \mathbb{R}^{n+1} \to \mathbb{R}$. Assume g is diff. w.r.t. first coordinate and continuous twice differentiable w.r.t all other coordinates [We also require $\frac{\partial^2 g}{\partial x_i \partial x_j}$ exists and continuous] then

$$\begin{aligned} g\left(t, B^{(1)}(t), B^{(2)}(t), ..., B^{(n)}(t)\right) &= g\left(0, B^{(1)}(0), ..., B^{(n)}(0)\right) \\ &= \int_0^t \frac{\partial g}{\partial X_o}\left(s, B^{(1)}(s), ..., B^{(n)}(s)\right) \, ds \\ &\quad + \sum_{i=1}^n \int_0^t \frac{\partial g}{\partial X_i}\left(s, B^{(1)}(s), ..., B^{(n)}(s)\right) \, dB^i(s) \\ &\quad + \frac{1}{2}\sum_{i=1}^n \int_0^t \frac{\partial^2 g}{\partial X_i^2}\left(s, B^{(1)}(s), ..., B^{(n)}(s)\right) \, ds \end{aligned}$$

8.4 Fact

Let $(M_t)_{t\ge0}$ be a continuous square-integrable martingale (with $M_0 = 0$). There is a unique continuous increasing process N_t that satisfies:

$$N_0 = 0 \ \ M_t^2 - N_t \ \text{ is a martinagle}$$

Moreover $N_t = \langle M \rangle_t$.

8.5 Proposition

Let $(B_t)_{t\geq 0}$ be a Brownian motion. Let $u(t,w)$ be a progressively meas. function s.t

$$E\left[\int_0^t u(s,w)^2 ds\right] < \infty \ \ \forall\, t \geq 0$$

Then

$$\langle \int_0^t u_s dB_s \rangle_t = \int_0^t u_s^2 ds$$

Proof:

Since $\int_0^t u(s,w)dB_s =: X_t$ is continuous, increasing and satisfies $X_0 = 0$. By the Proposition, it suffices to show that

$$X_t^2 - \int_0^t u(s,w)^2 ds$$

is a martingale. Indeed for $t > s$

$$\begin{aligned} E\left[X_t | F_s\right] - X_s &= E\left[X_t - X_s | F_s\right] \\ &= E\left[\int_s^t u^2(r,w) dB_r | F_s\right] \\ &= E\left[\int_s^t u^2(r,w) dr | F_s\right] \\ &= E\left[\int_0^t u^2(r,w) dr - \int_0^s u^2(r,w) dr | F_s\right] \end{aligned}$$

Hence:

$$E\left[X_t - \int_0^t u^2(r,w) dr | F_s\right] = X_s - \int_0^s u^2(r,w) dr$$

and

$$X_t - \int_0^t u^2(r,w) dr$$

is a martingale. ■

8.6 Definition

A stochastic process X_t is called an ***Ito process w.r.t Browian motion*** *B_t if*

$$X_t = X_0 + \int_0^t a(s,w) ds + \int_0^t b(s,w) dB_s \qquad 0 < t \leq T$$

For some progressive measurable a, b s.t

$$E\left[\int_0^T b^2(s,w) ds\right] < \infty$$

$$P\left[\int_0^T |a| ds < \infty\right] = 1$$

Thus let X_t be an Ito process def. as above, $g \in C^2(\mathbb{R} \times \mathbb{R})$. Let $Y_t = g(t, X_t)$. Then

$$Y_t - Y_0 = \int_0^t \left(\dot{g} + a g' + \frac{1}{2} b^2 g''\right) ds + \int_0^t (b g')\, dB_t \qquad 0 \leq t \leq T$$

When

$$\dot{g} = \partial_t g \ , \ g' = \partial_x g \ , \ g^{''} = \partial_{xx} g$$

Notation If

$$X_t = X_0 + \int_0^t a ds + \int_0^t b dB_s$$

We sometimes write $dX_t = ads + bdB_s$.

Proof: Recall for $f \in C^2(\mathbb{R})$

$$df(B_t) = f^{'}(B_s)dB_s + \frac{1}{2}f^{''}(B_s)d\langle B\rangle_t$$

$$dg(t, B_t) = \dot{g}dt + g'dX_t + \frac{1}{2}g^{''}d\langle X\rangle_t$$

Where $\langle X\rangle_t = \int_0^t b^2 ds$ (by the Proposition). ■

- $d\left(g(t, X_t)\right) = \dot{g}dt + g'dX_t + \frac{1}{2}g^{''}\left(dX_t\right)^2$ and when you expand $\left(dX_t\right)^2 = \left(adt + bdB_t\right)^2$
- follow the rule

$$dt \cdot dt = 0$$

$$dt \cdot dB_t = 0$$

Remark: One can similarly define multidimensional Ito processes and state an Ito's formula for them: If

$$(B_1, ..., B_d) \in \mathbb{R}^d$$

is a d-dimensional BM, an d-dimensional Ito-process is a collection of processes

$$X^{(1)}, ..., X^{(n)}$$

s.t

$$X_t^{(n)} = X_0^{(n)} + \int_0^t a_n ds + \int_{i=1}^d \int_0^t b_{n,i} dB_s$$

where $a_n, b_{n,i}$ satisfy appropriate conditions s.t. the integrals are well-defined. Ito's formula:

$$g\left(t, X_t^{(1)}, ..., X_t^{(n)}\right) - g\left(0, X_0^{(1)}, ..., X_0^{(n)}\right)$$

$$= \int_0^t \frac{\partial g}{\partial s}ds + \sum_{i=1}^n \int_0^t a_k \frac{\partial g}{\partial X_k}ds + \sum_{k=1}^n \sum_{j=1}^d b_{kj}\frac{\partial g}{\partial X_k}dB_j + \frac{1}{2}\sum_{k=1}^n \sum_{j=1}^d b_{kj}^2 \frac{\partial^2 g}{\partial X_k^2}ds$$