Exercise sheet 7

Stochastic Differential Equations, Girsanov and Repetition

Homework:
There are no more homeworks on this sheet!

Exercises for the tutorial:
Exercises will be discussed in the tutorials on 6th of February for all groups.

Exercise 7.1
Let $(B_t)_{t\geq 0}$ be standard Brownian motion and define for $\alpha, \beta > 0$ and for all $t \geq 0$

$$M_t = \exp\left(\alpha B_t - \left(\beta - \frac{\alpha^2}{2}\right) t\right).$$

(i) Find a stochastic differential equation, which is solved by M_t.

(ii) For which values of β is M_t a martingale?

(iii) Calculate $\mathsf{E}[\langle M \rangle_t]$ for $t \geq 0$ and β such that M_t is a martingale.

Exercise 7.2
Let $(B_t)_{t\geq 0}$ be standard Brownian motion and let $a, b > 0$.

(i) Show that the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = \frac{b-X_t}{1-t}\mathrm{d}t + \mathrm{d}B_t & \text{for } 0 \leq t < 1 \\ X_0 = a \end{cases}$$

has a strong (and therefore also a weak) solution on $[0, t]$ for all $0 \leq t < 1$.

(ii) Verify that

$$X_t = a(1-t) + bt + (1-t)\int_0^t \frac{\mathrm{d}B_s}{1-s}$$

solves the equation for all $0 \leq t < 1$.

Exercise 7.3
Let $(B_t)_{t\geq 0}$ be standard Brownian motion.

Please turn the page!

(i) Show that the process $(X_t)_{t\geq 0}$ defined by

$$\mathrm{d}X_t = \sin\left(\frac{1}{B_t}\right)\mathrm{d}t + \mathrm{d}B_t$$

is also Brownian motion.

(ii) Show that the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = \sin\left(\frac{1}{X_t}\right)\mathrm{d}t + \mathrm{d}B_t \\ X_0 = x_0 \end{cases}$$

has a weak solution on every interval $[0,T]$, $T \geq 0$.

Old exam questions:
Questions taken from last years final exams.

Exercise 7.4
Let $(B_t)_{0\leq t\leq 1}$ and $(B'_t)_{0\leq t\leq 1}$ be two independent Brownian motions. Which of the following statements is correct:

☐ $\mathbb{P}(B_1 > 1) < \mathbb{P}(B_5 > 1)$

☐ $\mathbb{P}(B_1 > 1) = \mathbb{P}(B_5 > 1)$

☐ $\mathbb{P}(B_1 > 1) > \mathbb{P}(B_5 > 1)$

Let P_1 be the distribution of $(B_t)_{0\leq t\leq 1}$, let P_2 be the distribution of $(2B_t)_{0\leq t\leq 1}$, and let P_3 be the distribution of $(B_t + B'_t)_{0\leq t\leq 1}$. Then
☐ ... all three are mutually singular.
☐ ... all three are mutually absolutely continuous
☐ ... P_1 and P_2 are mutually absolutely continuous and P_3 is singular w.r.t P_2.
☐ ... None of the above.

Let $X_t = B_t - tB_1$. Then
☐ $\mathbb{C}\mathrm{ov}(X_{0.25}, X_{0.75}) > 0$.
☐ $\mathbb{C}\mathrm{ov}(X_{0.25}, X_{0.75}) = 0$.
☐ $\mathbb{C}\mathrm{ov}(X_{0.25}, X_{0.75}) < 0$.

$\int_0^1 B_t \mathrm{d}B_t$
☐ ... is almost surely always negative.
☐ ... is almost surely always positive.
☐ ... admits almost surely negative and positive values.

Let $X_t = \int_0^t B_s \mathrm{d}s$. Then X_t
☐ ... is a martingale.
☐ ... is not a martingale. But the law of the process $(X_t)_{t\geq 0}$ is absolutely continuous with respect to the law of the process $(B_t)_{t\geq 0}$.
☐ None of the above.

Let $X_t = \int_0^t B_s \mathrm{d}B_s$.
☐ There exist finite constants $c, C > 0$ such that $ct < E[|X_t|] < Ct$ for all $t \geq 0$.
☐ There exist finite constants $c, C > 0$ such that $c\sqrt{t} < E[|X_t|] < C\sqrt{t}$ for all $t \geq 0$.
☐ There exist finite constants $c, C > 0$ such that $ct^2 < E[|X_t|] < Ct^2$ for all $t \geq 0$.

Exercise 7.5
Let $(B_t)_{t\geq 0}$ be a Brownian motion.

(i) Prove that almost surely,

$$\int_0^1 B_s \mathrm{d}B_s \geq -\frac{1}{2}.$$

(ii) Prove or disprove that almost surely,

$$\int_0^1 B_s \mathrm{d}B_s > -\frac{1}{2}.$$

Exercise 7.6
Let $(B_t)_{t\geq 0}$ be a Brownian motion. Show that

$$E[\sin B_t] = 0, \forall t \geq 0.$$

Exercise 7.7
Let $(B_t)_{t\geq 0}$ be a Brownian motion. Let

$$M_t = \int_0^t \frac{1}{1+s} \mathrm{d}B_s.$$

(i) Show that the $\lim_{t\to\infty} M_t$ exists almost surely.

(ii) What is the distribution of the almost sure limit?

Exercise 7.8
Let $(B_t)_{t\geq 0}$, $(B'_t)_{t\geq 0}$ be two independent Brownian motions and set $\mathcal{F}_t = \sigma(B_s; 0 \leq s \leq t)$ and $\mathcal{G}_t = \sigma(B_s, B'_s; 0 \leq s \leq t)$.

(i) Let $X_t = B_t B_{1-t}$, $0 \le t \le 1$. Prove or disprove that X_t is a martingale with respect to $\mathcal{F}_t$.

(ii) Let $Y_t = B_t B'_t$, $0 \le t \le 1$. Show that Y_t is a martingale with respect to $\mathcal{G}_t$.

Exercise 7.9

Let $(B_t)_{t\ge 0}$ be a Brownian motion. Consider the stochastic differential equation

$$\begin{cases} \mathrm{d}X_s = \mathrm{d}B_s - \frac{B_s}{1-s}\mathrm{d}s, \\ X_0 = 0. \end{cases}$$

Prove that for every $0 \le t < 1$, there exists a weak solution in the interval $[0, t]$.