Exercise sheet 4

Stopping Times and Optional Stopping Theorem

Homework:
Please hand in the solutions of Exercise 4.1 – 4.3 in the exercise class (for group 1) or lecture (for groups 2, 3 and 4) on 5th of December.

Exercise 4.1 (6 Points)
Let $(B_t)_{t\geq 0}$ be Brownian motion and consider a process $(X_t)_{0\leq t\leq 1}$ where

$$X_t := B_t - tB_1 \quad \text{for } 0 \leq t \leq 1.$$

(i) Show that $(X_t)_{0\leq t\leq 1}$ is a Gaussian process

(ii) Compute the covariance $\mathbb{C}\mathrm{ov}(X_s, X_t)$ for $0 \leq s, t \leq 1$.

(iii) Show that $(X_t)_{0\leq t\leq 1}$ does not have independent increments and is not time-homogeneous (i.e. the law of $(X_{t+s} - X_s)$ is not the same as $(X_t - X_0)$ for all $0 \leq s \leq t \leq 1$ such that $s + t \leq 1$).

Exercise 4.2 (7 Points)
Consider some measurable space $(\Omega, \mathcal{F})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$. For a stopping time T we have defined

$$\mathcal{F}_T := \{A \in \mathcal{F} : \ A \cap \{T \leq t\} \in \mathcal{F}_t \ \forall t \geq 0\}.$$

(a) (i) Show that $\mathcal{F}_T$ is a σ-field.

(ii) Show that $\mathcal{F}_S \subseteq \mathcal{F}_T$ holds for two stopping times S, T with $S \leq T$.

(b) Show that $S \wedge T = \min\{S, T\}$ and $S \vee T = \max\{S, T\}$ are also stopping times if S and T are stopping times.

(c) Prove that if T_n is a sequence of stopping times that increases to T, then T is a stopping time. Also prove that if T_n is a sequence of stopping times that decreases to T and $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, then T is a stopping time. (Why do you need right-continuity in this case?)

Theorem 1 (Optional Stopping Theorem) *Suppose $(X_t)_{t\geq 0}$ is a continuous martingale and $0 \leq S \leq T$ are stopping times. If the process $(X_t)_{t\geq 0}$ is dominated by an integrable random variable Y, i.e. $|X_{t\wedge T}| \leq Y$ almost surely, for all $t \geq 0$, then*

$$\mathsf{E}[X_T|\mathcal{F}_S] = X_S \textit{ almost surely.}$$

Please turn the page!

Exercise 4.3 (7 Points)
Let $(B_t)_{t\geq 0}$ be Brownian motion.

(i) Use the optional stopping theorem for a martingale that you know to show that, with $\tau_a = \inf\{t \geq 0 : B_t = a\}$,

$$E_0[e^{-\lambda\tau_a}] = e^{-a\sqrt{2\lambda}}, \quad \text{for all } \lambda, a > 0.$$

(ii) Show that, with $\tau_{-a} = \inf\{t \geq 0 : B_t = -a\}$, we have

$$E_0[e^{-\lambda\tau_a}] = E_0[e^{-\lambda\tau_a}\,\mathrm{I}_{\{\tau_a<\tau_{-a}\}}] + E_0[e^{-\lambda\tau_{-a}}\,\mathrm{I}_{\{\tau_{-a}<\tau_a\}}]e^{-2a\sqrt{2\lambda}}.$$

(iii) Use the previous results to show that $\tau = \tau_a \wedge \tau_{-a}$ satisfies

$$E_0[e^{-\lambda\tau}] = \operatorname{sech}\left(a\sqrt{2\lambda}\right),$$

where $\operatorname{sech}(x) = 2/(e^x + e^{-x})$.

Exercises for the tutorial:
Exercises 4.4 - 4.6 will be discussed in the tutorials on 5th (Groups 1,2) and 19th of November (Group 4). There is no lecture or tutorials on 12th of December.

Exercise 4.4
Let $(B_t)_{t\geq 0}$ be Brownian Motion.

(i) If $\tau_1 = \inf\{t > 0 : B_t > 0\}$, then $\mathsf{P}(\tau_1 = 0) = 1$.

(ii) If $\tau_2 = \inf\{t > 0 : B_t = 0\}$, then $\mathsf{P}(\tau_2 = 0) = 1$.

Exercise 4.5
Consider a Brownian motion $(B_t)_{t\geq 0}$ and the random time

$$T_{a,b} := \inf\{t \geq 0 : B_t \in \{a, b\}\}$$

for $a < 0 < b$. Show that

$$\mathsf{P}(B_{T_{a,b}} = a) = \frac{b}{|a| + b} \quad \text{and} \quad \mathsf{P}(B_{T_{a,b}} = b) = \frac{|a|}{|a| + b}.$$

Exercise 4.6
Find a stochastic process $(X_t)_{t\geq 0}$ and a stopping time T such that $(X_t)_{t\geq 0}$ is a martingale, but the stopped process $(X_{t\wedge T})_{t\geq 0}$ is not.