Final Exam

Monday, 24 February 2014, 15:30 - 16:30

Last name:

First name:

Date of birth:

Matriculation number:

Signature:

Problem	**1**	**2**	**3**	**4**	**5**	**Total**
Points	10	10	20	10	10	60
Points achieved						

Remarks:

(i) Please fill in the blanks on this page **completely** and **readably**.

(ii) Please start a new page for every new problem.

(iii) Except for a writing utensil, any kind of auxiliary material is prohibited.

(iv) You have to provide **detailed reasons** for every step in your mathematical arguments.

Notification of grades and exam inspection:
Grades and the date of the exam inspection (Klausureinsicht) will be published on the website of the chair.

In all the exercises, let $(B_t)_{t\geq 0}$ be standard Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ its canonical filtration.

Problem 1
Let $f : [0,\infty) \mapsto \mathbb{R}$, $t \to f(t)$ and define the process $(X_t)_{t\geq 0}$ by $X_t = B_t^3 + f(t)B_t$ for all $t \geq 0$. Find a function $f(t)$ such that (X_t) satisfies the martingale property and prove that $(X_t)_{t\geq 0}$ is a martingale in this case.

Problem 2
Fix $1 < T < \infty$. Decide whether each of the following is a well defined Itô integral or not. Justify your answer.

$$\text{(i)} \int_{1/2}^{1} B_{\frac{1}{t}} \mathrm{d}B_t \qquad \text{(ii)} \int_{1}^{T} B_{\frac{1}{t}} \mathrm{d}B_t \qquad \text{(iii)} \int_{1}^{\infty} B_{\frac{1}{t}} \mathrm{d}B_t.$$

Problem 3

(i) Let $(X_t)_{t\geq 0}$ be the process given by $X_t = e^{-B_t - \frac{1}{2}t}$ for all $t \geq 0$. Find two functions $\alpha(x,t)$ and $\beta(x,t)$ such that

$$\mathrm{d}X_t = \alpha(B_t,t)\mathrm{d}B_t + \beta(B_t,t)\mathrm{d}t$$

and calculate $\mathsf{E}[X_t]$ and $Var(X_t)$.

(ii) Show that, for $T < \infty$, there exists a weak solution $(Y_t)_{t\in[0,T]}$ of

$$\begin{cases} \mathrm{d}Y_t = \mathrm{d}B_t + \sqrt{Y_t}\mathrm{d}t & \text{for all } t \in [0,T] \\ Y_0 = 0. \end{cases}$$

Hint: remember that $\int_0^t B_s \mathrm{d}s \sim \mathcal{N}(0, \frac{t^3}{3})$.

Problem 4
Let $X = \max\{B_t : 0 \leq t \leq 1\}$ and set $Y := B_2 - B_1$.

(i) Prove that $P(X > Y) \geq 1/2$.

(ii) Calculate the distribution functions of X and of $Z := |Y|$.

(iii) Prove or disprove that $P(X > |Y|) = 1/2$.

Problem 5
Show that there exists an infinite sequence of independent random variables, which are $\mathcal{N}(0,1)$-distributed (normally distributed with mean zero and variance one) and measurable with respect to $\mathcal{F}_1 := \sigma(B_t : 0 \leq t \leq 1)$.