TU München
Zentrum Mathematik
Lehrstuhl für Wahrscheinlichkeitstheorie

Stochastic Analysis
Winter Term 2012/13
Berger/Guo/Kochler

Exercise sheet 4

Please hand in the solutions of Exercise 4.3 – 4.6 in the lecture on 13th December.

Exercises for the tutorial:

Exercise 4.1
Consider a Brownian motion $(B_t)_{0\le t\le 1}$.

(i) Are the laws of $(B_t)_{0\le t\le 1}$ and $(B_t+x)_{0\le t\le 1}$ $(x \neq 0)$ equivalent or mutually singular? Explain why.

(ii) Are the laws of $(B_t)_{0\le t\le 1}$ and $(B_t - t\cdot B_1)_{0\le t\le 1}$ $(x \neq 0)$ equivalent or mutually singular? Explain why.

Exercise 4.2
Consider a Brownian motion $(B_t)_{t\ge 0}$ and the random time

$$T_{a,b} := \inf\{t \ge 0 : \ B_t \in \{a, b\}\}$$

for $a < 0 < b$. Show that

$$\mathsf{P}\left(B_{T_{a,b}} = a\right) = \frac{b}{|a| + b} \quad \text{and} \quad \mathsf{P}\left(B_{T_{a,b}} = b\right) = \frac{|a|}{|a| + b}.$$

Homework:

Definition:
A stochastic process $(Z_t)_{t\ge 0}$ is a Gaussian process if for all $0 \le t_1 < t_2 < \ldots < t_n$ the vector $(Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n})$ is a Gaussian random vector, i.e. has a (multivariate) normal distribution.

Exercise 4.3 (6 Points)
Consider a Brownian motion $(B_t)_{t\ge 0}$.

(i) Show that $(B_t)_{t\ge 0}$ is a Gaussian process.

(ii) Consider the process $(X_t)_{0\le t\le 1}$ where

$$X_t := B_t - t\cdot B_1 \quad \text{for } 0 \le t \le 1.$$

Show that $(X_t)_{0\le t\le 1}$ is also a Gaussian process.

Please turn the page!

(iii) Compute the covariance $\mathsf{Cov}(X_s, X_t)$ for $0 \leq s, t \leq 1$.

Remark: The process $(X_t)_{0\leq t\leq 1}$ is called Brownian bridge.

Exercise 4.4 (5 Points)
Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$. Show that

(i) $\left(\sum_{i=1}^{n} \frac{X_i}{i}\right)_{n\in\mathbb{N}}$ is bounded in L^2;

(ii) $\sum_{i=1}^{\infty} \frac{X_i}{i}$ converges a.s. and in L^2.

Definition
A point $x \in (0, \infty)$ is called a *local point of increase* of a function $f : [0, \infty) \to \mathbb{R}$ if there exists an open interval (u, v) containing x such that $f(x) \geq f(a)$ for all $a \in (u, x)$ and $f(x) \leq f(b)$ for all $b \in (x, v)$.

Theorem (We omit the proof.):

(i) Almost surely, a Brownian motion $(B_t)_{t\geq 0}$ has no local point of increase.

(ii) Almost surely, a Brownian motion $(B_t)_{t\geq 0}$ has infinitely many local maxima.

Exercise 4.5 (5 Points)
Let $(B_t)_{t\geq 0}$ be a Brownian motion and set $X_t := B_t + t$ for $t \geq 0$. Answer the following questions and explain why:

(i) Compute the quadratic variation of $(X_t)_{t\geq 0}$.

(ii) How many local points of increase does $(X_t)_{t\geq 0}$ have?

(iii) Does $(X_t)_{t\geq 0}$ have infinitely many local maxima?

Exercise 4.6 (4 Points)
Let $(B_t)_{0\leq t\leq 1}$ be a Brownian motion. Are the laws of $(B_t)_{0\leq t\leq 1}$ and $(2B_t)_{0\leq t\leq 1}$ equivalent or mutually singular? Explain why.