Exercise sheet 3

Please hand in the solutions of Exercise 3.2 – 3.5 in the lecture on 22nd November.

Exercises for the tutorial:

Definition:
Consider a probability space $(\Omega, \mathcal{A}, \mathsf{P})$ and a filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e. an increasing family of sub-σ-algebras (i.e. $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{A}$ for $0 \leq s \leq t$).
A family of random variables $(M_t)_{t\geq 0}$ is said to be a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if the following conditions are fulfilled:

(i) M_t is measurable with respect to $\mathcal{F}_t$ for $t \geq 0$.

(ii) M_t is integrable for $t \geq 0$.

(iii) We have
$$\mathsf{E}[M_t|\mathcal{F}_s] = M_s \quad \text{a.s. for } 0 \leq s < t.$$

Exercise 3.1
Consider a Brownian motion $(B_t)_{t\geq 0}$ and the associated canonical filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e.

$$\mathcal{F}_t := \sigma(B_s :\ 0 \leq s \leq t) \quad \text{for } t \geq 0.$$

Show that the following stochastic processes are martingales with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$:

(i) $(B_t)_{t\geq 0}$

(ii) $(B_t^2 - t)_{t\geq 0}$

(iii) Fix $\alpha \in \mathbb{R}$ and consider $(M_t)_{t\geq 0}$ where

$$M_t := \exp\left(\alpha B_t - \frac{1}{2}\alpha^2 t\right).$$

Homework:

Exercise 3.2 (5 Points)
Consider a random variable X_1 with a $\mathcal{N}(0,1)$ distribution. Let Y be another random variable which is independent of X_1 and for which we have $\mathsf{P}(Y = 1) = \frac{1}{2} = \mathsf{P}(Y = -1)$. Further we define $X_2 := Y \cdot X_1$. Show that the following statements hold:

(i) X_2 has a $\mathcal{N}(0,1)$ distribution.

(ii) X_1 and X_2 are uncorrelated but not independent.

(iii) (X_1, X_2) does not have a two-dimensional normal distribution.

Please turn the page!

Exercise 3.3 (5 Points)
Consider a Brownian motion $(B_t)_{t\geq 0}$ and let us define $\mathcal{F}_t := \sigma(B_s :\ 0 \leq s \leq t)$.

(i) Show that $\mathcal{F}_s = \sigma(\mathcal{M}_s)$, where

$$\mathcal{M}_s := \Big\{ \{B_{t_1} \in A_1,\ B_{t_2} - B_{t_1} \in A_2,\ \ldots, B_{t_n} - B_{t_{n-1}} \in A_n\} :\ n \in \mathbb{N},\ 0 \leq t_1 < t_2 < \ldots < t_n \leq s,\ A_1,\ A_2,\ \ldots,\ A_n \in \mathcal{B}(\mathbb{R}) \Big\}.$$

(ii) For $s > 0$, show that $(B_{t+s} - B_s)_{t\geq 0}$ is independent of $\mathcal{F}_s$.

Exercise 3.4 (4 + 3 Points)
For a partition $E : 0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n = 1$ we define

$$V(E) := \sum_{i=0}^{n-1} \big[B(t_{i+1}) - B(t_i)\big]^2 \qquad \text{and} \qquad \Delta(E) := \max_{0\leq i\leq n-1} (t_{i+1} - t_i).$$

Prove the following statements:

(i) Almost surely there is a (random, depending on the Brownian Motion) refining sequence of partitions $(E_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} V(E_n) = 0$.

(ii*) Almost surely there is a (random, depending on the Brownian Motion) refining sequence of partitions $(E_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} V(E_n) = 0$ and $\lim_{n\to\infty} \Delta(E_n) = 0$.

Exercise 3.5 (6 Points)
Let $(E_n)_{n\in\mathbb{N}}$ be a refining sequence of partitions. Show that $\big(V(E_n)\big)_{n\in\mathbb{N}}$ is an inverse martingale, i.e.

$$\mathsf{E}\big[V(E_n)\big|V(E_{n+1}), V(E_{n+2}), \ldots\big] = V(E_{n+1}).$$

Help for the proof:

(i) Without loss of generality we may assume that for all $n \in \mathbb{N}$ the partition E_{n+1} contains only one additional partition point compared to E_n. Why?

(ii) For $0 < t_1 < t_2 < t_3$, consider

$$\mathcal{A}_-(t_2) := \sigma\big((B(s) - B(r))^2 :\ 0 \leq r \leq s \leq t_2\big),$$
$$\mathcal{A}_+(t_2) := \sigma\big((B(s) - B(r))^2 :\ t_2 \leq r \leq s\big)$$

and show that

$$\mathsf{E}\Big[\big(B(t_2) - B(t_1)\big) \cdot \big(B(t_3) - B(t_2)\big)\Big|\mathcal{A}_-(t_2), \mathcal{A}_+(t_2)\Big] = 0$$

holds.

(iii) Show

$$\mathsf{E}\big[V(E_n)\big|V(E_{n+1}), V(E_{n+2}), \ldots\big] = V(E_{n+1}).$$