Exercise sheet 2

Please hand in the solutions of Exercise 2.3 – 2.6 in the lecture on 8th November.

Exercises for the tutorial:

Exercise 2.1
Let $(B_t)_{t\geq 0}$ be a Brownian motion. Decide which of the following statements is true and mark it with a cross:

☐ $\mathsf{P}(B_1 > 1) < \mathsf{P}(B_5 > 1)$

☐ $\mathsf{P}(B_1 > 1) = \mathsf{P}(B_5 > 1)$

☐ $\mathsf{P}(B_1 > 1) > \mathsf{P}(B_5 > 1)$

Exercise 2.2
Let $(B_t)_{t\geq 0}$ be a Brownian motion.

(i) Compute the distribution of $4B_{\frac{1}{2}} - 3B_2 + 4s$ for some $s \in \mathbb{R}$.

(ii) Compute $\mathsf{E}\big[(B_3)^2 \big| B_1, B_2\big]$.

Homework:

Exercise 2.3 (4 Points)
Consider a random variable Z with a $\mathcal{N}(0,1)$-distribution. For $u \geq 1$, show that

$$\mathsf{P}(|Z| \geq u) \leq \sqrt{\frac{2}{\pi}} \cdot \exp\left(-\frac{u^2}{2}\right)$$

holds.

Exercise 2.4 (6 Points)
Let $(X_t)_{t\in[0,1]}$ be a stochastic process with continuous paths on some measurable space $(\Omega, \mathcal{F})$. Show that the mapping

$$\begin{aligned} [0,1] \times \Omega &\to \mathbb{R} \\ (t,\omega) &\mapsto X_t(\omega) \end{aligned}$$

is $\big(\mathcal{B}([0,1]) \otimes \mathcal{F}\big) - \mathcal{B}(\mathbb{R})$-measurable.

Exercise 2.5 (6 Points)
Let $(B_t)_{t\geq 0}$ be a Brownian motion and $a > 0$. Show that $(\frac{1}{\sqrt{a}} B_{at})_{t\geq 0}$ and $(B_{a+t} - B_a)_{t\geq 0}$ are Brownian motions.

Please turn the page!

Exercise 2.6 (4 Points)
Let $(B_t)_{t\geq 0}$ be a Brownian motion and $\sigma \in \mathbb{R}$. Compute

$$\mathsf{E}\big[\exp\big(\sigma(B_t - B_s)\big)\big], \qquad t > s \geq 0.$$