Exercise sheet 1

Please hand in the solutions of Exercise 1.3 and 1.4 in the lecture on 25th October.

Exercises for the tutorial:
At first recall the following definition:

Definition
A random vector $\vec{X} := (X_1, X_2, \ldots, X_n)^{tr}$ has a (multivariate) normal distribution iff

$$\langle (X_1, X_2, \ldots, X_n), (a_1, a_2, \ldots, a_n) \rangle = \sum_{i=1}^{n} a_i X_i$$

has a one-dimensional normal distribution for all $a_1, a_2, \ldots, a_n \in \mathbb{R}$. We write

$$\vec{X} \ \sim \ \mathcal{N}(\vec{\mu}, \Sigma)$$

where

$$\vec{\mu} := (\mathsf{E}[X_1], \mathsf{E}[X_2], \ldots, \mathsf{E}[X_n])^{tr}$$

is the mean vector of $\vec{X}$ and

$$\Sigma := \begin{pmatrix} \mathbb{C}\mathrm{ov}(X_1, X_1) & \mathbb{C}\mathrm{ov}(X_1, X_2) & \ldots & \mathbb{C}\mathrm{ov}(X_1, X_n) \\ \mathbb{C}\mathrm{ov}(X_2, X_1) & \mathbb{C}\mathrm{ov}(X_2, X_2) & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{C}\mathrm{ov}(X_n, X_1) & \ldots & \ldots & \mathbb{C}\mathrm{ov}(X_n, X_n) \end{pmatrix}$$

the covariance matrix of $\vec{X}$.

Exercise 1.1
Let $(B_t)_{0 \leq t \leq 1}$ be a Brownian motion. Compute the distribution of

$$\frac{B_t}{\sqrt{t}}$$

for $0 < t \leq 1$.

Exercise 1.2
Consider two random variables X and Y which are i.i.d. with distribution $\mathcal{N}(0, \sigma^2)$ for some $\sigma^2 > 0$. Show that $X + Y$ and $X - Y$ are also i.i.d. with distribution $\mathcal{N}(0, 2\sigma^2)$.

Homework:

Exercise 1.3 (6 Points)

(i) Consider a random vector

$$\vec{X} := \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

with a $\mathcal{N}(\vec{\mu}, \Sigma)$ distribution, some matrix $A \in \mathbb{R}^{k \times n}$ and some vector $\vec{b} \in \mathbb{R}^k$. Determine the distribution of $A\vec{X} + \vec{b}$.

Hint: You may use that the characteristic function $\varphi : \mathbb{R}^n \to \mathbb{C}$ of a n-dimensional $\mathcal{N}(\vec{\mu}, \Sigma)$-distribution is given by

$$\varphi(\vec{t}) = \exp\left(i \cdot \vec{t}^{\,tr} \cdot \vec{\mu} - \tfrac{1}{2} \cdot \vec{t}^{\,tr} \cdot \Sigma \cdot \vec{t}\right).$$

(ii) Consider a two-dimensional random vector (X_1, X_2) with common density

$$f(x_1, x_2) := \frac{\sqrt{2}}{\pi} \exp\left(-\frac{3}{2}x_1^2 - x_1 x_2 - \frac{3}{2}x_2^2\right).$$

Determine the marginal distributions of X_1 and X_2. Are X_1 and X_2 stochastically independent?

Hint: Recall the following fact about multivariate normal distributions: Consider a random vector $\vec{X} := (X_1, X_2, \ldots, X_n)^{tr}$ where $\vec{X} \sim \mathcal{N}(\vec{\mu}, \Sigma)$. If Σ is not singular, i.e. $det(\Sigma) > 0$ holds, then the inverse Σ^{-1} exists and $\vec{X}$ has the density

$$f_{\vec{X}}(\vec{x}) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \cdot \frac{1}{\sqrt{det(\Sigma)}} \cdot \exp\left(-\frac{1}{2}(\vec{x} - \vec{\mu})^{tr} \cdot \Sigma^{-1} \cdot (\vec{x} - \vec{\mu})\right)$$

with respect to the n-dimensional Lebesgue-measure.

Exercise 1.4 (4 Points)

Let $(B_t)_{0 \le t \le 1}$ be a Brownian motion. Compute

$$\mathsf{E}\left[(B_t)^k\right]$$

for $k \in \mathbb{N}$ and $0 \le t \le 1$.

Information:

- **Registration:** You will be able to sign up for the exercise classes via TUMonline starting from 18th October 2012, 18:00.
- **Website:** http://www-m14.ma.tum.de/en/teaching/ws12-13/stochastic-analysis/
- **Bonus system:** Please look up the details of the bonus system for the exams on our website.