Stochastic Analysis

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Contents

1 Definition and Construction of Brownian Motion

1.1 Historic origin

Brown (1827): Movement of a pollen in a liquid
Bachelier (1900): Model for stock market fluctuations
Einstein (1905): Motion of a particle

1.2 Heuristic description with symmetric random walks

$Y_1, Y_2, ...$ iid with $P[Y_i = 1] = \frac{1}{2} = P[Y_i = -1]$,
$S_k = \sum_{i=1}^{k} Y_i$, $k = 1, 2, ...$, $S_0 = 0$. Fix $N \in \mathbb{N}$
Rescale the process to the time interval $[0, 1]$:

$$X_{\frac{k}{N}} = \frac{1}{\sqrt{N}} S_k, \qquad k = 0, 1, 2 \ldots, N$$

Then,

(i) $X_0 = 0$.

(ii) For $0 \leq t_0 < t_1 < t_2 < ... < t_m \leq 1, t_i = \frac{k_i}{N}, k_i \in \{0, 1, ..., N\}, X_{t_i} - X_{t_{i-1}}$ are independent and $E[X_{t_i} - X_{t_{i-1}}] = 0$ and

$$Var(X_{t_i} - X_{t_{i-1}}) = Var(\frac{1}{\sqrt{N}} \sum_{j=k_{i-1}+1}^{k_i} Y_j) = \frac{1}{N}(k_i - k_{i-1}).$$

Due to the CLT, the laws of $(X_{t_i} - X_{t_{i-1}})$ converge to $N(0, t_i - t_{i-1})$ for $N \to \infty$ (and $\frac{k_i}{N} \xrightarrow[N \to \infty]{} t_i \in [0, 1]$).

This motivates the following definition:

1.3 Basic Definitions

Definition 1.1 Brownian Motion (BM) is a stochastic process $B_t(\omega), (0 \leq t \leq 1)$ on a probability space $(\Omega, \mathcal{A}, P)$ such that

(i) $B_0 = 0$ P-a.s.

(ii) For $0 \leq t_0 < t_1 < t_2 < ... < t_n \leq 1$, the increments $B_{t_i} - B_{t_{i-1}}$, $1 \leq i \leq n$, are independent with law $N(0, t_i - t_{i-1})$

(iii) $t \to B_t(\omega)$ is continuous for P-a.a.ω.

Definition 1.2 Let $(B_t)_{0 \leq t \leq 1}$ be a BM on the probability space $(\Omega, \mathcal{A}, P)$.
Then, the image measure of P under the map

$$\Omega \to C[0, 1]$$

$$\omega \to (B_t(\omega))_{0\le t\le 1}$$

is the **Wiener measure**.

The Wiener measure is a prob. measure on $(C[0,1],\mathcal{F})$, with $\mathcal{F}=\sigma(\{g_t : 0\le t\le 1\})$, where $g_t : C[0,1]\to\mathbb{R}, g_t(x)=x(t)$ $(x\in C[0,1])$.

Interpretation:
Def. 1.1: $t\to B_t(\omega)$ is a stochastic evolution in time.
Def. 1.2: $(B_t)_{0\le t\le 1}$ is a random variable with values in $C[0,1]$.

Theorem 1.3 *Brownian motion exists.*

There are different proofs of Theorem 1.3.
We give here a proof, which constructs "linear interpolations" on the sets $D_n=\{\frac{k}{2n} : 0\le k\le 2^n\}$. We follow the proof of Theorem 1.3. in [4].

Proof: Let $D=\bigcup_n D_n$ and let $(\Omega,\mathcal{A},P)$ be a probability space such that $\{Z_t, t\in D\}$ are iid random variables on $(\Omega,\mathcal{A},P)$, with law $N(0,1)$.
Let $B_0 := 0$ and $B_1 := Z_1$. For each $n\in\mathbb{N}$, we define the random variables $B_s, s\in D_n$ such that:

(1) For $r<s<t, r,s,t\in D_n, B_t-B_s$ is independent of B_s-B_r, and B_t-B_s has the law $N(0,t-s)$

(2) The vectors $(B_s, s\in D_n)$ and $(Z_t, t\in D\setminus D_n)$ are independent.

For $D_0=\{0,1\}$, we are done.
Proceeding inductively, assume that we followed the construction for some $n-1$. We then define B_s for $s\in D_n\setminus D_{n-1}$ by

$$B_s=\frac{1}{2}(B_{s-\frac{1}{2^n}}+B_{s+\frac{1}{2^n}})+\frac{1}{2^{\frac{n+1}{2}}}Z_s.$$

The first term is the linear interpolation of B at the neighboring points of s in D_{n-1}.

Therefore, B_s is independent of $(Z_t, t\in D\setminus D_n)$ and (2) is satisfied.
Moreover, since

$$\frac{1}{2}(B_{s+\frac{1}{2^n}}-B_{s-\frac{1}{2^n}})$$

depends only on $(Z_t, t \in D_{n-1})$, it is independent of $\frac{1}{2^{\frac{n+1}{2}}} Z_s$. By induction assumption, both terms have law $N(0, \frac{1}{2^{n+1}})$. Hence, their sums $B_s - B_{s-\frac{1}{2^n}}$ and their difference $B_{s+\frac{1}{2^n}} - B_s$ are iid with law $N(0, \frac{1}{2^n})$.

Exercise:
X and Y iid random variables with law $N(0, \sigma^2)$
$\Rightarrow X + Y, X - Y$ are iid random variables with law $N(0, 2\sigma^2)$.

To see that all increments $B_s - B_{s-\frac{1}{2^n}}, s \in D_n \setminus \{0\}$ are independent, it suffices to show that they are pairwise independent, since the vector of increments is Gaussian. We saw that $B_s - B_{s-\frac{1}{2^n}}$, $B_{s+\frac{1}{2^n}} - B_s$ (with $s \in D_n \setminus D_{n-1}$) are independent. The other possibility is that the increments are over intervals separated by some $s \in D_{n-1}$. Choose $s \in D_j$ with this property and j minimal, so that the two intervals are contained in $[s - \frac{1}{2^j}, s]$ and $[s, s + \frac{1}{2^j}]$.
By induction hypothesis, the increments over these two intervals of length $\frac{1}{2^j}$ are independent, and the increments over the intervals of lengths $\frac{1}{2^n}$ are constructed from the independent increments $B_s - B_{s-\frac{1}{2^j}}$ and $B_{s+\frac{1}{2^j}} - B_s$, respectively, using disjoint sets of random variables $(Z_t, t \in D_n)$.
Hence they are independent $\Rightarrow$ (1) is satisfied. This completes the induction.

Now, we interpolate between the dyadic points. More precisely, let

$$f_0(t) = \begin{cases} Z_1 & t = 1 \\ 0 & t = 0 \\ \text{linear in between.} \end{cases}$$

and for each $n \geq 1$,

$$f_n(t) = \begin{cases} 2^{-\frac{n+1}{2}} Z_t & t \in D_n \setminus D_{n-1} \\ 0 & t \in D_{n-1} \\ \text{linear between consecutive points in } D_n \end{cases}$$

$f_0, f_1, f_2, ...$are continuous functions and, $\forall n$ and $s \in D_n$

$$B_s := \sum_{j=0}^{n} f_j(s) = \sum_{j=0}^{\infty} f_j(s). \tag{1.1}$$

We prove (1.1) by induction. (1.1) holds for $n = 0$. Suppose it holds for $n-1$. Let $s \in D_n \setminus D_{n-1}$. Since for $0 \le j \le n-1$, the function f_j is linear on $[s - \frac{1}{2^n}, s + \frac{1}{2^n}]$ we get

$$\sum_{j=0}^{n-1} f_j(s) = \sum_{j=1}^{n-1} \frac{f_j(s - \frac{1}{2^n}) + f_j(s + \frac{1}{2^n})}{2} = \frac{1}{2}(B_{s-\frac{1}{2^n}} + B_{s+\frac{1}{2^n}}).$$

Since $f_n(s) = \frac{1}{2^{\frac{n+1}{2}}} Z_s$, this gives (1.1).
Since $Z_s \stackrel{d}{=} N(0,1)$, we have for $c > 1$, and n large enough,

$$P[|Z_1| \ge c\sqrt{n}] \le e^{-\frac{c^2 n}{2}}$$

(since $\int_x^\infty e^{-\frac{u^2}{2}} du \le \frac{1}{x} e^{-\frac{x^2}{2}}$, Proof: exercise).
$\Rightarrow$ the series

$$\sum_{n=0}^{\infty} P[\exists s \in D_n \text{with} |Z_s| \ge c\sqrt{n}] \le \sum_{n=0}^{\infty} \sum_{s \in D_n} P[|Z_s| \ge c\sqrt{n}] \le \sum_{n=0}^{\infty} (2^n + 1) e^{-\frac{c^2 n}{2}}$$

converges if $c > \sqrt{2 \log 2}$. Fix $c > \sqrt{2 \log 2}$.
Apply the Borel-Cantelli lemma :
$\exists N_0(\omega) < \infty$ s.t. for $n \ge N_0(\omega)$, and $s \in D_n$ we have $|Z_s| < c\sqrt{n}$
$\Rightarrow$For $n \ge N_0(\omega)$, $||f_n||_\infty < c\sqrt{n}\frac{1}{2^{\frac{n}{2}}}$
$\Rightarrow$For P-a.a. ω, the sequence $B_t^m = \sum_{n=0}^m f_n(t)$ converges uniformly in $t \in [0,1]$ for $m \to \infty$
$\Rightarrow B_t := \lim_{m\to\infty} B_t^m$ is continuous in t.

We check that the increments of B have the right finite-dimensional distributions:
Assume $0 \le t_1 < t_2 < ... < t_n \le 1$. Then, we find $0 \le t_{1,k} \le t_{2,k} \le ... \le t_{n,k} \le 1$ with $t_{i,k} \in D$ and $\lim_{k\to\infty} t_{i,k} = t_i$ and since $t \to B_t$ is continuous for P-a.a. ω, $B_{t_{i+1}} - B_{t_i} = \lim_{k\to\infty} (B_{t_{i+1,k}} - B_{t_{i,k}})$ P-a.s.

Since

$$\lim_{k\to\infty} E[B_{t_{i+1,k}} - B_{t_{i,k}}] = 0 \quad \text{and}$$

$$\lim_{k\to\infty} Cov(B_{t_{i+1,k}} - B_{t_{i,k}}, B_{t_{j+1,k}} - B_{t_{j,k}}) = \lim_{k\to\infty} I_{\{i=j\}}(t_{i+1,k} - t_{i,k}) = I_{\{i=j\}}(t_{i+1} - t_i),$$

the increments $B_{t_{i+1}} - B_{t_i}, i = 1, 2, ..., n$, are independent Gaussian random variables with mean 0 and the variance $t_{i+1} - t_i$, using Lemma 1.4 (s. below)

Lemma 1.4 $(X_n)_{n\in\mathbb{N}}$ *sequence of Gaussian random vectors and* $\lim_{n\to\infty} X_n = X$ *P-a.s.. If* $b := \lim_{n\to\infty} E[X_n]$ *and* $C := \lim_{n\to\infty} Cov(X_n)$ *exist, then* X *is Gaussian with mean* b *and Covariance Matrix* C.

Proof: See [4], Prop. 12.15. ☻

Definition 1.5 A stochastic process $(B_t)_{t\geq 0}$ on some prob. space $(\Omega, \mathcal{A}, P)$ is a Brownian Motion if:

(i) $B_0 = 0$ P-a.s.

(ii) For $0 \leq t_0 < t_1 < ... < t_n$, the increments $B_{t_i} - B_{t_{i-1}}$ are independent with law $N(0, t_i - t_{i-1})$

(iii) $t \to B_t(\omega)$ is continuous for P-a.a. ω.

We obtain $(B_t)_{t\geq 0}$ from a sequence of iid BMs $(B_t)_{0\leq t\leq 1}$ as follows:

$$B_t = B^{\lfloor t\rfloor}_{t-\lfloor t\rfloor} + \sum_{i=0}^{\lfloor t\rfloor -1} B^i_1 \qquad (t \geq 0)$$

i.e. by glueing the paths $(B^i_t)_{0\leq t\leq 1}$ together.

Then $(B_t)_{t\geq 0}$ is a BM. Proof: exercise

Definition 1.6 A stochastic process $(V_t)_{t\geq 0}$ is a Gaussian process if for all $t_1 < t_2 < ... < t_n$, the vector $(V_{t_1}, ..., V_{t_n})$ is a Gaussian random vector.

$(B_t)_{t\geq 0}$ is a Gaussian process.
Proof: See exercises.

2 Some properties of Brownian Motion

The paths of BM are random fractals in the follows sense:

Lemma 2.1 (Scaling invariance)
Let $(B_t)_{t\geq 0}$ be a BM and let $a > 0$. Then the process $(X_t)_{t\geq 0}$ given by $X_t = \frac{1}{a} B_{a^2 t} \quad (t \geq 0)$ is also a BM.

Proof: Independence and stationarity of the increments and continuity of the paths persist under the scaling. It remains to show that $X_t - X_s = \frac{1}{a}(B_{a^2t} - B_{a^2s})$ has the law $N(0, t-s)$. But $X_t - X_s$ is a Gaussian RV with the expectation 0 and variance $\frac{1}{a^2}E[(B_{a^2t} - B_{a^2s})^2] = \frac{1}{a^2}a^2(t-s) = t - s$. ☻

Example 2.2 (1) Let $a < 0 < b$ and consider $T_{a,b}$ with

$$T_{a,b} := \inf\{t \geq 0 : B_t \in \{a, b\}\}$$

the first exit time of BM from the intervall (a, b).
Then, with $X_t = \frac{1}{|a|}B_{a^2t}$,

$$E[T_{a,b}] = a^2 E[\inf\{t \geq 0 : X_t \in \{-1, \frac{b}{|a|}\}\}] = a^2 E[T_{-1,\frac{b}{|a|}}]$$

In particular $E[T_{b,b}] = b^2 E[T_{-1,1}] = const \cdot b^2$.

(2) Ruin probabilities:

$$P[(B_t)_{t\geq 0} \text{ exits } (a,b) \text{ at } a] = P[(X_t)_{t\geq 0} \text{ exits } (-1, \frac{b}{|a|}) \text{ at } -1]$$

depends only on the ratio $\frac{b}{|a|}$.

Theorem 2.3 (Time inversion)
Let $(B_t)_{t\geq 0}$ be a BM. Then the process $(X_t)_{t\geq 0}$ given by

$$X_t = \begin{cases} 0 & t = 0 \\ tB_{\frac{1}{t}} & t > 0 \end{cases}$$

is again a BM.

Proof: Recall that $(B_{t_1}, ..., B_{t_n}), 0 \leq t_1 < t_2 < ... \leq t_n$ are Gaussian random vectors and are therefore characterized by their expectations and their Covariances

$$\text{Cov}(B_{t_i}, B_{t_j}) = t_i \wedge t_j \tag{2.1}$$

Proof of (2.1): Let $t_i < t_j$. Then

$$E[B_{t_i}B_{t_j}] = E[B_{t_i}(B_{t_j} - B_{t_i})] + E[B_{t_i}^2] = 0 + t_i = t_i$$

$(X_t)_{t\geq 0}$ is also a Gaussian process (check!) and the Gaussian random vectors $(X_{t_1}, ..., X_{t_n})$ have expectations $E[X_{t_i}] = 0, 1 \leq i \leq n$. For $t > 0, h \geq 0$, the Covariance of X_t and X_{t+h} is given by

$$\text{Cov}(X_t, X_{t+h}) = \text{Cov}(tB_{\frac{1}{t}}, (t+h)B_{\frac{1}{t+h}}) = t(t+h)\text{Cov}(B_{\frac{1}{t}}, B_{\frac{1}{t+h}}) = t$$

Hence, the laws of all the finite vectors $(X_{t_1}, ..., X_{t_n}), 0 \leq t_1 < t_2 < ... \leq t_n$, are the same as for BM. The paths $t \mapsto X_t$ are clearly continuous for all $t > 0$ (for P-a.a. ω).
For $t = 0$, we use the following two facts:

(1) Since $\mathbb{Q}$ is countable, $(X_t, t \geq 0, t \in \mathbb{Q})$ has the same law as $(B_t, t \geq 0, t \in \mathbb{Q})$
$\Rightarrow \lim_{t \searrow 0, t \in \mathbb{Q}} X_t = 0$ P-a.s.

(2) $\mathbb{Q} \cap (0, \infty)$ is dense in $(0, \infty)$ and $(X_t)_{t\geq 0}$ is continuous on $(0, \infty)$ (for P-a.a. ω) so that $0 = \lim_{t \searrow 0, t \in \mathbb{Q}} X_t = \lim_{t \to 0} X_t = 0$ P-a.s.

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Example 2.4 (Ornstein-Uhlenbeck-process)
Let $(B_t)_{t\geq 0}$ be a BM, and set $X_t = e^{-t} B_{e^{2t}}, (t \in \mathbb{R})$.
Then $X_t \stackrel{d}{=} N(0,1) \quad \forall t$
Proof: X_t is a Gaussian RV with $E[X_t] = 0, \text{Var}(X_t) = e^{-2t} e^{2t} = 1$ ☻

Further $(X_{-t})_{t \in \mathbb{R}}$ has the same law as $(X_t)_{t \in \mathbb{R}}$.

Proof: Set $\widetilde{X}_t = X_{-t}, (t \in \mathbb{R})$ and

$$\widetilde{B}_t = \begin{cases} tB_{\frac{1}{t}} & t > 0 \\ 0 & t = 0. \end{cases}$$

Then $\widetilde{X}_t = e^t B_{e^{-2t}} = e^t \widetilde{B}_{e^{2t}} e^{-2t} = e^{-t} \widetilde{B}_{e^{2t}}$.
Since $(\widetilde{B}_t)_{t\geq 0}$ is a BM, $(\widetilde{X}_t)_{t \in \mathbb{R}} \stackrel{d}{=} (X_t)_{t \in \mathbb{R}}$. ☻

$(X_t)_{t \in \mathbb{R}}$ is a Gaussian process with $E[X_t] = 0, \forall t$ and

$$\text{Cov}(X_s, X_t) = E[X_s, X_t] = e^{-(s+t)} E[B_{e^{2s}} B_{e^{2t}}] \stackrel{(2.1)}{=} e^{-(s+t)} e^{2(s \wedge t)} = e^{-|t-s|}.$$

Later we will see that $(\frac{1}{\sqrt{2}} X_t)_{t \geq 0}$ is a (weak) solution of the stochastic differential equation

$$dX_t = dB_t - X_t dt.$$

Corollary 2.5 (Law of large numbers)
$(B_t)_{t\geq 0}$ BM. Then,

$$\lim_{t \to \infty} \frac{1}{t} B_t = 0 \qquad P\text{-a.s.}$$

Remark:

$$\lim_{n\to\infty} \frac{1}{n} B_n = 0 \qquad \text{P-a.s.}$$

since $B_n = \sum_{i=1}^n (B_i - B_{i-1}) = \sum_{i=1}^n Y_i$, $(Y_i)_{i\geq 1}$ iid with law $N(0,1)$.

Proof: Let

$$X_t = \begin{cases} tB_{\frac{1}{t}} & t > 0 \\ 0 & t = 0. \end{cases}$$

Then $\lim\limits_{t\to\infty} \frac{1}{t} B_t = \lim\limits_{t\to\infty} X_{\frac{1}{t}} = X_0 = 0$ P-a.s. ☻

Remark 2.6 The law of the iterated logarithm says that

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t \log\log t}} = 1 \qquad \text{P-a.s.} \tag{2.2}$$

$$\liminf_{t\to\infty} \frac{B_t}{\sqrt{2t \log\log t}} = -1 \qquad \text{P-a.s.} \tag{2.3}$$

In particular,

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{t}} = +\infty, \liminf_{t\to\infty} \frac{B_t}{\sqrt{t}} = -\infty \qquad \text{P-a.s.}$$

Theorem 2.7 (Paley, Wiener, Zygmund)
Let $(B_t)_{t\geq 0}$ be a BM. Then

$$P[\{\omega : t \mapsto B_t(\omega) \textit{is nowhere differentiable}\}] = 1$$

Proof: Assume that there is $t_0 \in [0,1]$ such that $t \mapsto B_t(\omega)$ is diff. in t_0. Then, there is a constant $M < \infty$ such that

$$\sup_{s\in[0,1]} \left| \frac{B_{t_0+s} - B_{t_0}}{s} \right| \leq M \tag{2.4}$$

If $t_0 \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ for some $n > 2, k \leq 2^n$, then we have for $1 \leq j \leq n$,

$$\left| B_{\frac{k+j}{2^n}} - B_{\frac{k+j-1}{2^n}} \right| \leq M(2j+1)\frac{1}{2^n} \tag{2.5}$$

Proof of (2.5):

$$\begin{aligned} \left| B_{\frac{k+j}{2^n}} - B_{\frac{k+j-1}{2^n}} \right| &\leq \left| B_{\frac{k+j}{2^n}} - B_{t_0} \right| + \left| B_{t_0} - B_{\frac{k+j-1}{2^n}} \right| \\ &\leq M\frac{j+1}{2^n} + M\frac{j}{2^n} \\ &\leq M(2j+1)\frac{1}{2^n}. \end{aligned}$$

Let $A_{n,k} \subseteq C[0,1]$ be the collection of functions satisfying (2.5) for $j = 1,2,3$.
Claim:

$$P[A_{n,k}] \le P[|B_1| \le \frac{7M}{\sqrt{2^n}}]^3 \tag{2.6}$$

Proof of (2.6):

$$\frac{B_{k+j}}{2^n} - \frac{B_{k+j-1}}{2^n} \overset{d}{=} N(0, \frac{1}{2^n}) \overset{d}{=} \frac{1}{\sqrt{2^n}} B_1$$

and $\frac{B_{k+j}}{2^n} - \frac{B_{k+j-1}}{2^n}, j = 1,2,3$ are independent. Further,

$$P\left[|B_1| \le \frac{7M}{\sqrt{2^n}}\right]^3 \le \left(\frac{7M}{\sqrt{2^n}}\right)^3.$$

Hence,

$$P\left[\bigcup_{k=1}^{2^n} A_{n,k}\right] \le 2^n \left(\frac{7M}{\sqrt{2^n}}\right)^3 = \frac{(7M)^3}{\sqrt{2^n}}$$

$$\Rightarrow \sum_{n=2}^{\infty} P\left[\bigcup_{k=1}^{2^n} A_{n,k}\right] < \infty.$$

Therefore, using the Borel-Cantelli lemma,

$$\begin{aligned} P[(2.4) \text{ holds for some } t_0 \in [0,1]] &\le P\left[\bigcup_{k=1}^{2^n} A_{n,k} \quad \text{happens for infinitely n}\right] \\ &= P\left[\bigcap_{m=2}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{k=1}^{2^n} A_{n,k}\right] \\ &= 0. \end{aligned}$$

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Corollary 2.8 *For P-a.a. ω, the function $t \mapsto B_t(\omega)$ is **not** of bounded variation on **any** interval.*
Recall that for g which is right-continuous on $[a,b]$, we set

$$V_g[a,b] = \sup\left\{\sum_{k=1}^{m} |g(t_k) - g(t_{k-1})| : a \le t_0 < t_1 < ... < t_m \le b, m \in \mathbb{N}\right\}.$$

$V_g[a,b]$ is the variation of g on $[a,b]$. We say that g is of bounded variation (BV) on $[a,b]$ if $V_g[a,b] < \infty$.

Example:
If $t \mapsto g(t)$ is increasing on $[a,b]$, g is BV on $[a,b]$, and $V_g[a,b] = g(b) - g(a)$.

Lemma 2.9 *Assume g is BV on $[a,b]$ and right-continuous $\Rightarrow \exists g_1, g_2 : [a,b] \to \mathbb{R}, g_1, g_2$ increasing and right-continuous such that*

$$g = g_1 - g_2 \tag{2.7}$$

Proof: See literature.

Proof of Corollary 2.8: A theorem by Lebesgue says that an increasing function $g : [a, b] \to \mathbb{R}$ is differentiable for λ-a.a. $s \in [a, b]$. ☻

Remark 2.10 Let g be right-continuous and increasing on $[a, b]$. Then, g defines a measure ν_g on $[a, b]$ by

$$\nu_g((a_1, b_1]) = g(b_1) - g(a_1), \quad a \le a_1 < b_1 \le b.$$

If g is BV on $[a, b]$, and $f \in C[a, b]$ we can define

$$\int_a^b f(s)dg(s) := \int_a^b f(s)\nu_{g_1}(ds) - \int_a^b f(s)\nu_{g_2}(ds)$$

(with g_1, g_2 from (2.7)). For a BM $(B_t)_{t\ge 0}$, this procedure can not be applied - nevertheless, we will be able to define $\int_a^b f(s)dB_s$.

Remark 2.11 A continuous function g which is BV on [0,1] has **quadratic variation** 0, i.e. for $E_n \subseteq [0, 1], E_n = \{0, t_1, ..., t_n, 1\}, (0 \le t_1 < ... < t_n \le 1)$ we have

$$\sum_{t_i\in E_n} (g(t_{i+1}) - g(t_i))^2 \le \max_{t_i\in E_n} |g(t_{i+1}) - g(t_i)| \cdot \sum_{t_i\in E_n} |g(t_{i+1}) - g(t_i)| \underset{n\to\infty}{\longrightarrow} 0$$

if $s(E_n) \underset{n\to\infty}{\longrightarrow} 0$, where $s(E_n) := \sup_{t_i\in E_n} |t_{i+1} - t_i|$. is the **mesh** of E_n.

Theorem 2.12 *$(B_t)_{t\ge 0}$ BM. Let (E_n) be a sequence of partitions with $s(E_n) \underset{n\to\infty}{\longrightarrow} 0, E_n \subseteq E_{n+1} \subseteq E_{n+2} \subseteq ...$ Then, $\forall t > 0$,*

$$V_t^n := \sum_{\substack{t_i\in E_n,\\ t_{i+1}\le t}} (B_{t_{i+1}} - B_{t_i})^2 \underset{n\to\infty}{\longrightarrow} t \qquad P\text{-a.s. and in } L^2.$$

Proof:

(1) Convergence in L^2:

$$E[V_t^n] = \sum_{\substack{t_i\in E_n,\\ t_{i+1}\le t}} (t_{i+1} - t_i) \underset{n\to\infty}{\longrightarrow} t.$$

Using the independence of the increments,

$$\mathrm{Var}(V_t^n) = \sum_{\substack{t_i\in E_n,\\ t_{i+1}\le t}} \mathrm{Var}((B_{t_{i+1}} - B_{t_i})^2) \underset{(*)}{=} \sum_{\substack{t_i\in E_n,\\ t_{i+1}\le t}} 2(t_{i+1} - t_i)^2$$

$\Rightarrow \mathrm{Var}(V_t^n) \underset{n\to\infty}{\longrightarrow} 0$, see Remark 2.11

Proof of (*): $Y \overset{d}{=} N(0, \sigma^2)$

$$\Rightarrow \mathrm{Var}(Y^2) = E[Y^4] - E[Y^2]^2 = 3\sigma^4 - \sigma^4 = 2\sigma^4.$$

(2) We first show P-a.s. convergence along dyadic partitions $E_n = \{0, \frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n}, ..., 1\}$. For these partitions,

$$\text{Var}(V_t^n) = 2 \cdot 2^n \left(\frac{1}{2^n}\right)^2 = \frac{2}{2^n}$$

$\Rightarrow \sum_{n=1}^{\infty} \text{Var}(V_t^n) < \infty$.
Using Lemma 2.13 below we conclude that $V_t^n \underset{n\to\infty}{\longrightarrow} t$ P-a.s..

(3) For general sequence of partitions, use (2) and an approximation argument. ☻

Lemma 2.13 *$Y_1, Y_2, ...$ RVs with $\sum_{n=1}^{\infty}$ Var$(Y_n) < \infty$. Then, $(Y_n - E[Y_n]) \underset{n\to\infty}{\longrightarrow} 0$ P-a.s..*

<u>Proof:</u> See exercises.

3 The Cameron-Martin Theorem and the Paley-Wiener stochastic integral

We know several facts about paths of BM, for instance:

$$P[\exists t_0 \in [0,1] \text{ s.t. } t \to B_t \text{ differentiable in } t_0] = 0$$

Do these properties remain true for $(B_t + ct)_{0\le t\le 1}$ or, more generally, for $(B_t + h(t))_{0\le t\le 1}$ where $h \in C[0,1]$?

We denote by

$$H = \left\{ h \in C[0,1] : \text{ there is } f \in L^2[0,1] \text{ s.t. } h(t) = \int_0^t f(s)ds, 0 \le t \le 1 \right\}$$

the **Cameron-Martin space** (or Dirichlet space).
Given $h \in H$, f is uniquely determined as an element of $L^2[0,1]$ and we write

$$h' = f$$

Example:

$h' = f$ is true λ-a.s. (λ = Lebesgue measure on $[0,1]$).

Recall that for two measures μ, ν on $(\Omega, \mathcal{A})$ we write $\mu \perp \nu$ and say "μ and ν are singular" if there is $A \in \mathcal{A}$ with $\mu(A) = 0, \nu(A^c) = 0$.
We write $\nu \ll \mu$ and say "ν is absolutely continuous w.r.t. μ" if, $\forall A \in \mathcal{A}$, $\mu(A) = 0 \Rightarrow \nu(A) = 0$.
We write $\nu \approx \mu$ and say "ν and μ are equivalent" if $\nu \ll \mu$ and $\mu \ll \nu$.
Let μ be Wiener measure on $(C[0,1], \mathcal{F})$ and μ_h be the law of $(B_t + h(t))_{0 \le t \le 1}$.

Theorem 3.1 *Assume $h \in C[0,1]$ and $h(0) = 0$.*

(i) If $h \notin H$ then $\mu_h \perp \mu$.

(ii) If $h \in H$ then $\mu_h \approx \mu$.

For the proof, we will need the following quantity:

$$Q_n(h) := 2^n \sum_{j=1}^{2^n} \left(h\left(\frac{j}{2^n}\right) - h\left(\frac{j-1}{2^n}\right) \right)^2 \qquad (n = 1, 2, ...) \tag{3.1}$$

Lemma 3.2 $Q_n(h), n = 1, 2, ...$ *is an increasing sequence and*

$$h \in H \Leftrightarrow \sup_n Q_n(h) < \infty.$$

Moreover, if $h \in H$, then

$$Q_n(h) \underset{n\to\infty}{\longrightarrow} \int_0^1 h'(s)^2 ds = \int_0^1 f(s)^2 ds.$$

Proof: The general inequality $(a+b)^2 \le 2a^2 + 2b^2$ gives

$$\left[h\left(\frac{j}{2^n}\right) - h\left(\frac{j-1}{2^n}\right)\right]^2 \le 2\left[h\left(\frac{2j-1}{2^{n+1}}\right) - h\left(\frac{j-1}{2^n}\right)\right]^2 + 2\left[h\left(\frac{j}{2^n}\right) - h\left(\frac{2j-1}{2^{n+1}}\right)\right]^2$$

Summing this inequality over $j \in \{1, 2, ..., n\}$ gives $Q_n(h) \le Q_{n+1}(h) \Rightarrow Q_n(h)$ is increasing in n. For $h \in H$ with $h' = f$, we have, using Jensen's inequality

$$Q_n(h) = 2^n \sum_{j=1}^{2^n} \left(\int_{\frac{j-1}{2^n}}^{\frac{j}{2^n}} f(s) ds \right)^2 \le \sum_{j=1}^{2^n} \int_{\frac{j-1}{2^n}}^{\frac{j}{2^n}} f(s)^2 ds = \int_0^1 f(s)^2 ds$$

Hence, $h \in H \Rightarrow \sup_n Q_n(h) < \infty$.
Proof of "$\Rightarrow$": Let $t \sim \mathcal{U}([0,1])$, then there exists a sequence of intervals $I_n(t) = [a_n, b_n] = \left[\frac{k_n - 1}{2^n}, \frac{k_n}{2^n}\right]$ s.t. $t \in I_n(t), \forall n$. Given $I_1(t), ... I_n(t)$, the interval $I_{n+1}(t)$ is, with probability $\frac{1}{2}$ the left or the right half of $I_n(t)$. Let $M_n = M_n(t) = 2^n(h(b_n)) - h(a_n))$, then $(M_n)_{n\ge1}$ is a martingale w.r.t. $\sigma(I_n(t)$ (on $([0,1], B_{[0,1]}, \lambda)$). Furthermore:

$$E[M_n^2] = 2^{2n} \sum_{k=1}^{2^n} \left(h\left(\frac{k}{2^n}\right) - h\left(\frac{k-1}{2^n}\right) \right)^2 \cdot \frac{1}{2^n} = Q_n(h)$$

If $\sup_n Q_n(h) < \infty$, $(M_n)_{n\ge1}$ is a martingale which is bounded in L^2, i.e. $\sup_n E[M_n^2] < \infty$.

We prove later:

Lemma 3.3 *Assume $(M_n)_{n\geq 1}$ is a martingale on $(\Omega, \mathcal{A}, P)$ and $(M_n)_{n\geq 1}$ is bounded in L^2. Then there exists a RV X s.t.*

$$M_n \underset{n\to\infty}{\longrightarrow} X \qquad P\text{-a.s. and in } L^2$$

By Lemma 3.3, $M_n \to X$ a.s. and in L^2. Let

$$g(s) := \int_0^s X(t)dt$$

For j, m fixed, we have $\forall n$:

$$h\left(\frac{j}{2^m}\right) = \int_0^{\frac{j}{2^m}} M_n(t)dt \underset{n\to\infty}{\longrightarrow} \int_0^{\frac{j}{2^m}} X(t)dt$$

$\Rightarrow h\left(\frac{j}{2^m}\right) = g\left(\frac{j}{2^m}\right) \quad \forall j, m$ and by continuity, $g(s) = h(s) \quad \forall s \in [0,1]$ and $h'(s) = X(s) \quad \lambda$-a.s.
$\Rightarrow h \in H$ and

$$Q_n(h) = E[M_n^2] \underset{n\to\infty}{\longrightarrow} E[X^2] = \int_0^1 (h'(t))^2 dt.$$

☻

Proof of Lemma 3.3: M_n is bounded in L^1 and by the martingale convergence theorem (Theorem 14.13 in Probability Theory lecture notes) $M_n \underset{n\to\infty}{\longrightarrow} X$ P-a.s. and $X \in L^1$. We have for $m > n$:

$$E[(M_m - M_n)^2] = E[M_m^2] - E[M_n^2] \tag{3.2}$$

since

$$E[M_m M_n] = E[E[M_m M_n|\mathcal{A}_n]] = E[M_n E[M_m|\mathcal{A}_n]] = E[M_n^2]$$

Fatou's Lemma implies from (3.2)

$$E[(X - M_n)^2] \leq E\left[\liminf_{m\to\infty}(M_m - M_n)^2\right] \leq \liminf_{m\to\infty} E[M_m^2] - E[M_n^2]$$

The last expression tends to 0 a.s. for $n \to \infty$, since $E[M_n^2]$ is increasing in n:

$$E[M_n^2] \overset{(3.2)}{=} \sum_{k=2}^n E[(M_k - M_{k-1})^2] + E[M_1^2].$$

☻

Lemma 3.4 (The Paley-Wiener stochastic integral)
Let $(B_t)_{t\geq 0}$ be a BM and $h \in H$. Then

$$\xi_n := 2^n \sum_{j=1} 2^n \left(h\left(\frac{j}{2^n}\right) - h\left(\frac{j-1}{2^n}\right)\right)(B_{\frac{j}{2^n}} - B_{\frac{j-1}{2^n}})$$

converges a.s. and in L^2. The limit is denoted by $\int_0^1 h' dB$.

Proof: Recall from the construction of BM that

$$B_{\frac{2j-1}{2^n}} = \frac{1}{2}\left(B_{\frac{2j-2}{2^n}} + B_{\frac{2j}{2^n}}\right) + \sigma_n Z_{\frac{2j-1}{2^n}}$$

where $\sigma_n = 2^{-(n+1)/2}$ and Z_t are iid standard normal.
Therefore:

$$\xi_n - \xi_{n-1} = 2^n \sigma_n \sum_{j=1}^{2^{n-1}} \left(2h\left(\frac{2j-1}{2^n}\right) - h\left(\frac{2j-2}{2^n}\right) - h\left(\frac{2j}{2^n}\right)\right) Z_{\frac{2j-1}{2^n}}$$

which implies that $(\xi_n)_{n\geq 1}$ is a martingale.

$$(E[\xi_n|\mathcal{A}_{n-1}] = E[\xi_n - \xi_{n-1}|\mathcal{A}_{n-1}] + E[\xi_{n-1}|\mathcal{A}_{n-1}] = E[\xi_{n-1}|\mathcal{A}_{n-1}])$$

$$E[\xi_n^2] = 2^{2n} \sum_{j=1}^{2^n} \left(h\left(\frac{j}{2^n}\right) - h\left(\frac{j-1}{2^n}\right)\right)^2 E\left[\left(B_{\frac{j}{2^n}} - B_{\frac{j-1}{2^n}}\right)^2\right] = Q_n(h).$$

Hence, for $h \in H$, the convergence follows from Lemma 3.3. ☻

Proof of the Cameron-Martin Theorem: Let μ_n and $\mu_{h,n}$ denote the finite-dimensional distributions on the set D_n. Then the Radon-Nikodym derivative $\frac{d\mu_{h,n}}{d\mu_n}$ is the ratio of the two Lebesgue densities. For $x \in C[0,1]$ and $\Delta_j x = \Delta_j^{(n)} x = x\left(\frac{j}{2^n}\right) - x\left(\frac{j-1}{2^n}\right)$,

$$\frac{d\mu_{h,n}}{d\mu_n}(x) = \prod_{j=1}^{2^n} \exp\left(-\frac{(\Delta_j x - \Delta_j h)^2}{2^{1-n}}\right) \exp\left(\frac{(\Delta_j x)^2}{2^{1-n}}\right) = \exp(-H_n(x))$$

with $H_n(x) = 2^{n-1} \sum_{j=1}^{n} ((\Delta_j h)^2 - 2\Delta_j x \Delta_j h)$.
By Theorem 14.5 in the Probability Theory lecture notes, $\exp(-H_n(x))$ is a martingale under μ, since it is non-negative it converges a.s. to a finite RV X. We show later:

$$\mu_h(A) = \int_A X d\mu + \mu_h(A \cap \{X = \infty\})$$

for $A \in \mathcal{F}$. This implies:

$$\mu(X = 0) = 1 \quad \Rightarrow \quad \mu \perp \mu_h$$
$$\mu(X > 0) = 1 \quad \Rightarrow \quad \mu \ll \mu_h$$

We have $E_\mu[H_n] = \int H_n(x)\mu(dx) = \frac{1}{2}Q_n(h)$ and $\mathrm{Var}_\mu(H_n) = 2^{2n-2} \cdot 4 \cdot \sum_{j=1}^{2^n} (\Delta_j h)^2 \mathrm{Var}_\mu(\Delta_j x) = Q_n(h)$. By Chebyshev's inequality, we get

$$P_\mu\left(H_n \leq \frac{1}{4}Q_n(h)\right) = P_\mu\left(\frac{1}{2}Q_n(h) - H_n \geq \frac{1}{4}Q_n(h)\right) \leq \frac{Q_n(h)}{\left(\frac{1}{4}\mathbb{Q}_n(h)\right)^2} = \frac{16}{Q_n(h)}.$$

Now, if $h \notin H$, then by Lemma 3.2, $H_n \to \infty$ and $x = 0$ μ-a.s..
For the converse, suppose $h \in H$. By Lemma 3.4 (and the second part of Lemma 3.2),

$$H(x) \underset{n\to\infty}{\longrightarrow} \frac{1}{2}||h'||_2^2 - \int_0^1 h' dB.$$

Therefore $x > 0$ μ-a.s. and $\mu \ll \mu_h$. Finally note that $\mu_h \ll \mu \Leftrightarrow \mu \ll \mu_{-h}$. ☻

4 Brownian Motion as a continuous martingale

Definition 4.1 Consider a probability space $(\Omega, \mathcal{F}, P)$.

(i) A **filtration** is a sequence of σ-fields $(\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}, \quad \forall s < t$.

(ii) A stochastic process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ is **adapted** to $(\mathcal{F}_t)_{t\geq 0}$ if X_t is $\mathcal{F}_t$-measurable, $\forall t \geq 0$.

Suppose $(X_t)_{t\geq 0}$ is a stochastic process on $(\Omega, \mathcal{F}, P)$. Then we can define a filtration $(\mathcal{F}_t)_{t\geq 0}$ by taking $\mathcal{F}_t := \sigma(\{X_s, 0 \leq s \leq t\})$, i.e. $\mathcal{F}_t$ is the σ-field generated by $\{X_s, 0 \leq s \leq t\}$. Then, $(X_t)_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$.

Definition 4.2 A real-valued stochastic process $(X_t)_{t\geq 0}$ is a **martingale** with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if it is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and

$$E[|X_t|] < \infty \quad \forall t \geq 0 \tag{4.1}$$

and, for $0 \leq s \leq t$

$$E[X_t|\mathcal{F}_s] = X_s \quad \text{P-a.s.} \tag{4.2}$$

The process $(X_t)_{t\geq 0}$ is a **submartingale** with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if it is adapted to $(\mathcal{F}_t)_{t\geq 0}$, (4.1) holds and for $0 \leq s \leq t$

$$E[X_t|\mathcal{F}_s] \geq X_s \quad \text{P-a.s.} \tag{4.3}$$

and it is a **supermartingale** with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if it is adapted to $(\mathcal{F}_t)_{t\geq 0}$, (4.1) holds and, for $0 \leq s \leq t$

$$E[X_t|\mathcal{F}_s] \leq X_s \quad \text{P-a.s.} \tag{4.4}$$

Remark 4.3 If $(X_t)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$, $|X_t|$ is in general not a martingale but a submartingale. More generally, if $(X_t)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ and $f : \mathbb{R} \to \mathbb{R}$ is a convex function such that $E[|f(X_t)|] < \infty$, $\forall t \geq 0$, then $f(X_t)_{t\geq 0}$ is a submartingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Proof:

$$E[f(X_t)|\mathcal{F}_s] \overset{Jensen}{\geq} f(E[X_t|\mathcal{F}_s]) = f(X_s) \quad \text{P-a.s.}$$

hence (4.2) holds. ☻

Remark 4.4 Let $(B_t)_{t\geq 0}$ be a BM and $\mathcal{F}_t = \sigma(\{X_s, 0 \leq s \leq t\})$. Then $(B_t)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Proof:

$$E[B_t|\mathcal{F}_s] = E[B_t - B_s|\mathcal{F}_s] + E[B_s|\mathcal{F}_s] = 0 + B_s \quad \text{P-a.s.}$$

(See exercise 3.3 (ii): $B_t - B_s$ is independent of $\mathcal{F}_s$, hence $E[B_t - B_s|\mathcal{F}_s] = 0$.) ☻

Definition 4.5 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$. A RV T with values in $[0, \infty]$ is a **stopping time** with respect to $(\mathcal{F}_t)_{t\geq 0}$ if

$$\{T \leq t\} \in \mathcal{F}_t, \quad \forall t \geq 0.$$

Example 4.6 Let $(B_t)_{t\geq 0}$ be a BM and $\mathcal{F}_t = \sigma(\{X_s, 0 \leq s \leq t\})$, $t \geq 0$.

(i) Let $y \neq 0$ and $T = \inf\{t \geq 0 : B_t = y\}$. Then, T is a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$.
<u>Proof:</u>

$$\{T \leq t\} = \bigcap_{n=1}^{\infty} \bigcup_{s:s\in\mathbb{Q}\cap(0,t)} \{B_s \in U(y, \frac{1}{n})\} \in \mathcal{F}_t, \qquad \text{where}$$

$$U(y, \frac{1}{n}) := \{z \in \mathbb{R} : |z - y| < \frac{1}{n}\}$$

(ii) Let $I = (a, b)$, $0 < a < b$ and $T = \inf\{t \geq 0 : B_t \in I\}$. Then, T is not a stopping time because $\{T \leq t\} \notin \mathcal{F}_t$.
<u>Proof:</u> See [4].

Definition 4.7 Assume T is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Define

$$\mathcal{F}_T := \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}.$$

$\mathcal{F}_T$ is called **the σ-field of events observable until time T**.
A martingale $(X_t)_{t\geq 0}$ is a **continuous** martingale if $P[t \to X_t(\omega)$ is continuous$] = 1$.

Theorem 4.8 (Optional stopping)
Suppose $(X_t)_{t\geq 0}$ is a continuous martingale and $0 \leq S \leq T$ stopping times. If the process $(X_{t\wedge T})_{t\geq 0}$ is dominated by an integrable RV X, i.e. $|X_{t\wedge T}| \leq X$, $\forall t \geq 0$ a.s. and $E[|X|] < \infty$, then $E[X_T|\mathcal{F}_S] = X_S$ P-a.s.

<u>Proof:</u> This can be derived from the result for martingales in discrete time (see Theorem 14.9 in the Probability Theory lecture notes for the case $S = 0$). See [4] for details.

Theorem 4.9 (Wald's Lemma for BM)
Let $(B_t)_{t\geq 0}$ be a BM and T a stopping time such that either

(a) $E[T] < \infty$ or

(b) $(B_{t\wedge T})_{t\geq 0}$ is dominated by an integrable RV.

Then, we have $E[B_T] = 0$.

Remark 4.10 One does need a condition on T, as the following example shows:
Let $T = \inf\{t : B_t = 1\}$. (Then $T < \infty$ P-a.s., see exercise 5.2). Clearly, $E[B_T] = 1$. We conclude from Theorem 4.9 that $E[T] = \infty$ and that $(B_{t\wedge T})_{t\geq 0}$ is not dominated by an integrable RV.

Proof of Theorem 4.9: We show that (a) implies (b). Suppose $E[T] < \infty$, and define $M_k = \max_{0\le t\le 1} |B_{t+k} - B_k|$ and $M = \sum_{k=1}^{\lfloor T\rfloor} M_k$. Then

$$\begin{aligned} E[M] &= E[\sum_{k=1}^{\lfloor T\rfloor} M_k] \\ &= \sum_{k=1}^{\infty} E[I_{\{T>k-1\}} M_k] \\ &= \sum_{k=1}^{\infty} P[T > k-1] E[M_k] \\ &\le E[M_0]E[T+1] \end{aligned}$$

But $E[M_0] = E[\max_{0\le t\le 1} |B_t|] < \infty$ (Proof: exercise).
If (b) is satisfied, we can apply the optional stopping theorem (Theorem 4.8) with $S = 0$, giving $E[X_T|\mathcal{F}_0] = X_0$, P-a.s. which yields that $E[B_T] = 0$. ☻

5 Stochastic integrals with respect to Brownian Motion

Let $(B_t)_{t\ge 0}$ be a BM on some probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t$ the completion of $\sigma(\{B_s, s \le t\})$ (see Theorem 1.32 in the Probability Theory lecture notes). Then, $(B_t)_{t\ge 0}$ is adapted to $(\mathcal{F}_t)_{t\ge 0}$.

Definition 5.1 A process $\{X_t(\omega) : t \ge 0, \omega \in \Omega\}$ is **progressively measurable** if for each $t \ge 0$ the mapping $X : [0, t] \times \Omega \to \mathbb{R}$ is measurable with respect to the σ-field $\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t$.

Lemma 5.2 *Any process $(X_t)_{t\ge 0}$ which is adapted and either right-continuous or left-continuous is also progressively measurable.*

Proof: Assume $(X_t)_{t\ge 0}$ is right-continuous. Fix $t > 0$. For $n \in \mathbb{N}$, $0 \le s \le t$ define

$$X_0^{(n)}(\omega) = X_0(\omega)$$

and

$$X_s^{(n)}(\omega) = X_{\frac{(k+1)t}{2^n}}(\omega) \qquad \text{for } \frac{kt}{2^n} < s \le \frac{(k+1)t}{2^n} \qquad k = 0, 1, 2, ..., 2^n - 1.$$

The mapping $(s, \omega) \to X_s^{(n)}(\omega)$ is $\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t$-measurable.
By right-continuity, we have $\lim_{n\to\infty} X_s^{(n)}(\omega) = X_s(\omega)$ for all $s \in [0, t]$ and $\omega \in \Omega$, hence the limit mapping $(s, \omega) \to X_s(\omega)$ is also $\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t$-measurable.
The left-continuous case is analogous.

☻

We construct the integral by starting with "simple" progressively measurable processes $H_t(\omega)$ and then proceeding to more complicated ones. Consider first step processes $\{H_t(\omega) : t \geq 0, \omega \in \Omega\}$ of the form

$$H_t(\omega) = \sum_{j=1}^{k} A_j(\omega) I_{(t_j, t_{j+1}]}(t) \qquad \text{for } 0 \leq t_1 < \ldots < t_{k+1}$$

where A_j is $\mathcal{F}_{t_j}$-measurable, $1 \leq j \leq k$.

We define the integral as

$$\int_0^\infty H_s dB_s := \sum_{j=1}^{k} A_j (B_{t_{j+1}} - B_{t_j})$$

Now let H be a progressively measurable process satisfying $E\left[\int_0^\infty H_s^2 ds\right] < \infty$. Suppose H can be approximated by a sequence of progressively measurable step processes $H^{(n)}, n \geq 1$, then we define

$$\int_0^\infty H_s dB_s := \lim_{n\to\infty} \int_0^\infty H_s^{(n)} dB_s\,. \tag{5.1}$$

More precisely, let $||H||_2^2 := E\left[\int_0^\infty H_s^2 ds\right]$. We will show that

(1) Every progressively measurable H satisfying $E\left[\int_0^\infty H_s^2 ds\right] < \infty$ can be approximated in the $||\cdot||_2$ - norm by progressively measurable step processes.

(2) For each approximating sequence, the limit in (5.1) exists in the L^2-sense.

(3) This limit does not depend on the approximating sequence of step processes.

We start with (1):

Lemma 5.3 *For every progressively measurable process $\{H_s(\omega) : s \geq 0, \omega \in \Omega\}$ satisfying $E\left[\int_0^\infty H_s^2 ds\right] < \infty$ there exists a sequence $(H^{(n)})_{n\in\mathbb{N}}$ of progressively measurable step processes such that $\lim_{n\to\infty} ||H^{(n)} - H||_2 = 0$.*

Proof: We approximate the progressively measurable process successively by

- a bounded progressively measurable process
- a bounded, almost surely continuous progressively measurable process
- a progressively measurable step process.

Let $H = \{H_s(\omega), s \geq 0, \omega \in \Omega\}$ be a progressively measurable process with $||H||_2 < \infty$. First define

$$H_s^{(n)}(\omega) = \begin{cases} H_s(\omega) & s \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\lim_{n\to\infty} ||H^{(n)} - H||_2 = 0$.
Second, approximate any progressively measurable process H on a finite interval by truncating its values, i.e. define $H^{(n)}$ by

$$H_s^{(n)} = (H_s(\omega) \wedge n) \vee (-n).$$

Clearly, $H^{(n)}$ is progressively measurable and $||H^{(n)} - H||_2 \to 0$.
Now we approximate any uniformly bounded progressively measurable H by bounded, almost surely continuous progressively measurable processes.

Let $h = \frac{1}{n}$ and

$$H_s^{(n)}(\omega) = \frac{1}{h}\int_{s-h}^{s} H_t(\omega)dt$$

(where we set $H_s(\omega) = H_0(\omega)$ for $s < 0$). $H^{(n)}$ is again progressively measurable (since we only take averages over the past). $H^{(n)}$ is almost surely continuous. Further, $\forall \omega \in \Omega$ and almost every (with respect to Lebesgue measure) $s \in [0, t]$,

$$\lim_{h\to 0} \frac{1}{h}\int_{s-h}^{s} H_t(\omega)dt = H_s(\omega).$$

Since H is uniformly bounded, we obtain that $\lim_{n\to\infty} ||H^{(n)} - H||_2 = 0$.

Finally, a bounded, amost surely continuous, progressively measurable process H can be approximated by a sequence of progressively measurable step processes $H^{(n)}$ by taking

$$H_s^{(n)} = H\left(\frac{j}{n}, \omega\right) \qquad \text{for } \frac{j}{n} \le s \le \frac{j+1}{n}.$$

The process $H^{(n)}$ are again progressively measurable and one easily sees that

$$\lim_{n\to\infty} ||H^{(n)} - H||_2 = 0.$$

This completes the proof of Lemma 5.3. ☻

Lemma 5.4 *Let H be a progressively measurable step process and $E\left[\int_0^\infty H_s^2 ds\right] < \infty$. Then*

$$E\left[\left(\int_0^\infty H_s dB_s\right)^2\right] = E\left[\int_0^\infty H_s^2 ds\right]$$

Proof: Let $H = \sum_{i=1}^k A_i I_{(a_i, a_{i+1}]}$ be a progressively measurable step process. Then

$$E\left[\left(\int_0^\infty H_s dB_s\right)^2\right] = E\left[\sum_{i,j=1}^k A_i A_j (B_{a_{i+1}} - B_{a_i})(B_{a_{j+1}} - B_{a_j})\right]$$

$$= \sum_{i=1}^k E[A_i^2 (B_{a_{i+1}} - B_{a_i})^2] + 2\sum_{i=1}^k \sum_{j=i+1}^k E[A_i A_j (B_{a_{i+1}} - B_{a_i}) \cdot E[(B_{a_{j+1}} - B_{a_j})|\mathcal{F}_{a_j}]]$$

$$= \sum_{i=1}^k E[A_i^2] E[(B_{a_{i+1}} - B_{a_i})^2] = E\left[\int_0^\infty H_s^2 ds\right]$$

☻

Corollary 5.5 *Suppose $(H^{(n)})_{n\in\mathbb{N}}$ is a sequence of progressively measurable step processes such that*

$$E[\int_0^\infty (H_s^{(n)} - H_s^{(m)})^2 ds] \underset{n,m\to\infty}{\longrightarrow} 0.$$

Then

$$E[(\int_0^\infty (H_s^{(n)} - H_s^{(m)}) dB_s)^2] \underset{n,m\to\infty}{\longrightarrow} 0.$$

Proof: Because the difference of two progressively measurable step processes is again a progressively measurable step process, Lemma 5.4 can be applied to $H^{(n)} - H^{(m)}$ and yields the claim. ☻

We showed (1). The following theorem addresses (2) and (3).

Theorem 5.6 *Suppose $(H^{(n)})_{n\in\mathbb{N}}$ is a sequence of progressively measurable step processes and H a progressively measurable process such that*

$$\lim_{n\to\infty} E[\int_0^\infty (H_s^{(n)} - H_s)^2 ds] = 0$$

Then

$$\lim_{n\to\infty} \int_0^\infty H_s^{(n)} dB_s =: \int_0^\infty H_s dB_s$$

exists as a limit in the L^2-sense and does not depend on the choice of $(H^{(n)})_{n\in\mathbb{N}}$. Moreover, we have

$$E[\int_0^\infty (H_s^{(n)} - H_s^{(m)})^2 dB_s] \underset{n,m\to\infty}{\longrightarrow} 0 \tag{5.2}$$

Proof: By the triangle inequality $(H^{(n)})_{n\in\mathbb{N}}$ satisfies the assumptions of Corollary 5.5 and hence $\left(\int_0^\infty H_s^{(n)} dB_s\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in L^2. Since L^2 is complete, the limit exists and does not depend on the choice of the approximating sequence. Finally, (5.2) follows from Lemma 5.4, applied to $H^{(n)}$, taking the limit as $n \to \infty$. ☻
This completes the construction of the stochastic integral $\int_0^\infty H_s^2 dB_s$ for progressively measurable processes H with $E\left[\int_0^\infty H_s^2 ds\right] < \infty$.

Remark 5.7 If the sequence of step processes in Theorem 5.6 is chosen such that

$$\sum_{n=1}^\infty E\left[\int_0^\infty (H_s^{(n)} - H_s)^2 ds\right] < \infty,$$

then by (5.2) we get

$$\sum_{n=1}^\infty E\left[\left(\int_0^\infty (H_s^{(n)} - H_s) dB_s\right)^2\right] < \infty$$

and therefore, almost surely,

$$\sum_{n=1}^\infty \left(\int_0^\infty H_s^{(n)} B_s - \int_0^\infty H_s dB_s\right)^2 < \infty.$$

This implies that, almost surely,

$$\lim_{n\to\infty}\int_0^\infty H_s^{(n)}dB_s = \int_0^\infty H_s dB_s.$$

We now want to describe the stochastic integral as a process in time. We will see that it will be a continuous martingale.

Definition 5.8 Suppose $H = \{H_s(\omega) : s \geq 0, \omega \in \Omega\}$ is progressively measurable with $E\left[\int_0^\infty H_s^2 ds\right] < \infty$. Define the progressively measurable process $\{H_s^t(\omega), s \geq 0, \omega \in \Omega\}$

$$H_s^t(\omega) := H_s(\omega)I_{\{s\leq t\}}.$$

Then, the stochastic integral of H up to time t is defined

$$\int_0^t H_s dB_s := \int_0^\infty H_s^t dB_s.$$

Remark 5.9 We have seen already in Lemma 3.4 that for $g \in L^2[0,1]$ we can define $\int_0^1 g(s)dB_s$. Provided that both integrals exist, the Paley-Wiener integral from Lemma 3.4 agrees with the stochastic integral just defined.

Proof: See exercises.

Definition 5.10 A stochastic process $(X_t)_{t\geq 0}$ is a **modification** of a stochastic process $(Y_t)_{t\geq 0}$ if, for every $t \geq 0$, we have $P[X_t = Y_t] = 1$.

Theorem 5.11 *Assume that* $(H_s(\omega))_{s\geq 0}$ *is progressively measurable and* $E\left[\int_0^t H_s(\omega)^2 ds\right] < \infty, \quad \forall t \geq 0$. *Then there exists a modification* $(M_t)_{t\geq 0}$ *of* $\left(\int_0^t H_s dB_s\right)_{t\geq 0}$ *such that*

$$P[t \mapsto M_t(\omega) \text{ is continuous}] = 1.$$

Further, $(M_t)_{t\geq 0}$ *is a martingale and hence*

$$E\left[\int_0^t H_s dB_s\right] = 0 \quad \forall t \geq 0.$$

Proof: Fix $t_0 \in \mathbb{N}$ and let $H^{(n)}$ be a sequence of progressively measurable step processes such that

$$||H^{(n)} - H^{t_0}||_2 \underset{n\to\infty}{\longrightarrow} 0$$

$$\Rightarrow E\left[\left(\int_0^\infty (H_s^{(n)} - H_s^{t_0})dB_s\right)^2\right] \underset{n\to\infty}{\longrightarrow} 0.$$

For $s \leq t$ the random variable $\int_0^s H_u^{(n)}dB_u$ is $\mathcal{F}_s$-measurable and $E[\int_s^t H_u^{(n)}dB_u|\mathcal{F}_s] = 0$ (proof: exercise!) which implies that the process $\left(\int_0^t H_u^{(n)}dB_u\right)_{0\leq t\leq t_0}$ is a martingale, $\forall n$. By Doob's maximal inequality, see below, for $p = 2$,

$$E\left[\sup_{0\leq t\leq t_0}\left(\int_0^t H_s^{(n)}dB_s - \int_0^t H_s^{(m)}dB_s\right)^2\right] \leq 4E\left[\left(\int_0^{t_0}(H_s^{(n)} - H_s^{(m)})dB_s\right)^2\right] \underset{n\to\infty}{\longrightarrow} 0.$$

This implies that $M_t^{(n)} := \int_0^t H_s^{(n)} dB_s, 0 \leq t \leq t_0,\ n = 0, 1, 2, \dots$ defines a Cauchy sequence in the space of continuous functions on $[0, t_0]$ (equipped with the supremum norm). We denote the limit of this Cauchy sequence by $(M_t)_{0\leq t\leq t_0}$. Hence, the process $(M_t)_{0\leq t\leq t_0}$ is almost surely a uniform limit of continuous processes and therefore almost surely continuous. Due to Theorem 5.6,

$$P\left[M_t = \int_0^t H_s dB_s\right] = 1.$$

For fixed $t \in [0, t_0]$, the random variable $\int_0^t H_s dB_s$ is the limit (in L^2) of $\int_0^t H_s^{(n)} dB_s$, hence it is $\mathcal{F}_t$-measurable, and $\int_t^{t_0} H_s dB_s$ has conditional expectation $E[\int_t^{t_0} H_s dB_s|\mathcal{F}_t] = 0$. Therefore, $\int_0^t H_s dB_s$ is a conditional expectation of M_{t_0}, given $\mathcal{F}_t$, i.e.

$$M_t = E\left[\int_0^{t_0} H_s dB_s|\mathcal{F}_t\right].$$

Therefore, $(M_t)_{0\leq t\leq t_0}$ is a martingale, as a process of successive predictions, (see (14.4) in Probability Theory lecture notes). ☻

Doob's maximal inequality
Suppose $(X_t)_{t\geq 0}$ is a continuous martingale and $p > 1$. Then, for any $t \geq 0$

$$E[\sup_{0\leq s\leq t} |X_s|^p] \leq \left(\frac{p}{p-1}\right)^p E[|X_t|^p].$$

Proof: See literature.

6 Itô's formula and examples

Let $f \in C^1(\mathbb{R})$ (f continuously differentiable) and $x : [0, \infty) \to \mathbb{R}$ x continuous and BV on $[0, t]$. Then

$$f(x(t)) - f(x(0)) = \int_0^t f'(x(s))dx(s)$$

Example:
$x(s) = s, \forall s \geq 0$. Then $f(t) - f(0) = \int_0^t f'(s)ds$. Itô's formula gives an analogue for the case when x is replaced by a BM B_t. The crucial difference is that the second derivative of f is needed.

Theorem 6.1 (Itô's formula I)
Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable such that $E[\int_0^t (f'(B_s))^2 ds] < \infty$ for some $t > 0$, where $(B_t)_{t\geq 0}$ is a BM. Then, almost surely for all $s \in [0, t]$

$$f(B_s) - f(B_0) = \int_0^s f'(B_u)dB_u + \frac{1}{2}\int_0^s f''(B_u)du.$$

Example 6.2 Let $f(x) = x^2$. We have $E[\int_0^s B_u^2 du] = \int_0^s u du < \infty, \forall s > 0$. Hence

$$B_s^2 = 2\int_0^s B_u dB_u + s \Rightarrow B_s^2 - s = 2\int_0^s B_u dB_u.$$

We conclude from Theorem 5.11 that $(B_s^2 - s)_{s\geq 0}$ is a martingale.

Example 6.3 Let $f(x) = x^3$. We have $E[\int_0^s B_u^4 du] = \int_0^s E[B_u^4] du < \infty, \forall s > 0$. Hence

$$B_s^3 = 3\int_0^s B_u^2 dB_u + 3\int_0^s B_u du\,.$$

Let

$$M_s = B_s^3 - 3\int_0^s B_u du, \quad s \geq 0.$$

We conclude from Theorem 5.11 that $(M_s)_{s\geq 0}$ is a martingale.

To prove Theorem. 6.1, we will need the following.

Theorem 6.4 *Let $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t > 0$, and $0 = t_1^{(n)} < ... < t_n^{(n)} = t$ are partitions such that their mesh*

$$\max_{1\leq i\leq n-1} |t_{i+1}^{(n)} - t_i^{(n)}| \to 0.$$

Then

$$\sum_{j=1}^{n-1} f(B_{t_j^{(n)}})(B_{t_{j+1}^{(n)}} - B_{t_j^{(n)}})^2 \to \int_0^t f(B_s) ds$$

Proof: For $f \equiv 1$, see Theorem 2.12. For general f, see Theorem 7.12 in Mörter/Peres.
Proof of Theorem 6.1: We write $w(\delta, M)$ for the modulus of continuity of f'' on $[-M, M]$:

$$w(\delta, M) = \sup_{s,t\in[-M,M],|s-t|<\delta} |f''(s) - f''(t)|$$

Using Taylor's formula, for any $x, y \in [-M, M], x < y$ with $|x - y| < \delta$,

$$|f(y) - f(x) - f'(x)(y - x) + \frac{1}{2}f''(x)(y - x)^2| \leq w(\delta, M)(y - x)^2.$$

Take a sequence $0 = t_1^{(n)} < ... < t_n^{(n)} = t$. We write $0 = t_1 < ... < t_n = t$ for simplicity. With

$$\delta_B := \max_{1\leq i\leq n-1} |B_{t_{i+1}} - B_{t_i}|$$

and

$$M_B := \max_{0\leq s\leq t} |B_s|,$$

we get

$$|\sum_{i=1}^{n-1}(f(B_{t_{i+1}}) - f(B_{t_i})) - \sum_{i=1}^{n} f'(B_{t_i})(B_{t_{i+1}} - B_{t_i}) - \sum_{i=1}^{n-1} \frac{1}{2} f''(B_{t_i})(B_{t_i+1} - B_{t_i})^2|$$

$$\leq w(\delta_B, M_B) \sum_{i=1}^{n-1} (B_{t_{i+1}} - B_{t_i})^2$$

Now, $\sum_{i=1}^{n-1} f(B_{t_{i+1}}) - f(B_{t_i}) = f(B_t) - f(B_0)$, and there is a sequence of partitions with mesh going to 0 s.t.

$$\sum_{i=1}^{n-1} f'(B_{t_i})(B_{t_{i+1}} - B_{t_i}) \to \int_0^t f'(B_s) dB_s \qquad P-a.s.$$

$$\sum_{i=1}^{n-1} f''(B_{t_i})(B_{t_{i+1}} - B_{t_i})^2 \to \int_0^t f''(B_s) ds \qquad P-a.s.$$

$$\sum_{i=1}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \to t \qquad P-a.s.$$

By continuity of the Brownian path, $w(\delta_B, M_B)$ converges almost surely to 0.
This proves Itô's formula for fixed t, or indeed almost surely for all $s \in \mathbb{Q} \cap [0,t]$. Since all the terms in Itô's formula are continuous almost surely, we get the result simultaneously for all $s \in [0,t]$. ☻

We next state Itô's formula for functions f which can depend also on time.

Theorem 6.5 (Itô's formula II)
Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x,t) \mapsto f(x,t)$ be twice continuously differentiable in the x-coordinate and once continuously differentiable in the t-coordinate. Assume that $E[\int_0^t (\partial_x f(B_s,s))^2 ds] < \infty$ for some $t > 0$. Then, almost surely for all $s \in [0,t]$:

$$f(B_s, s) - f(B_0, 0) = \int_0^s \partial_x f(B_u, u) dB_u + \int_0^s d_t f(B_u, u) du + \frac{1}{2} \int_0^s \partial_{xx} f(B_u, u) du$$

<u>Proof:</u> See Mörter/Peres.

Example 6.6 Fix $\alpha > 0$.

$$f(B_t, t) = e^{\alpha B_t - \frac{1}{2}\alpha^2 t} = M_t \qquad (M_0 = 1)$$

Then

$$f(B_s, s) - 1 = \int_0^s \alpha M_u dB_u + \int_0^s (-\frac{1}{2}\alpha^2) M_u du + \frac{1}{2} \int_0^s \alpha^2 M_u du$$

$$\Rightarrow M_s - M_0 = \int_0^s \alpha M_u dB_u.$$

We conclude from Theorem 5.11 that $(M_t)_{t \geq 0}$ is a martingale (details: exercise). M solves the stochastic differential equation

$$"dM_s = \alpha M_s dB_s,\ M_0 = 1"$$

which can be written in integral form

$$M_s = 1 + \int_0^s M_u dB_u,\ s \geq 0.$$

Definition 6.7 $(B_t)_{t\geq 0} = (B_t^{(1)}, ..., B_t^{(d)})_{t\geq 0}$ is a d-dim. BM if $(B_t^{(1)})_{t\geq 0}, ..., (B_t^{(d)})_{t\geq 0}$ are iid one-dimensional BMs. For $H_s = (H_s^{(1)}, ..., H_s^{(d)})$ we write

$$\int_0^t H_s dB_s = \sum_{i=1}^d \int_0^t H_s^{(i)} dB_s^{(i)}.$$

Theorem 6.8 (Multidimensional Itô formula)
Let $(B_t)_{t\geq 0}$ be a d-dim. BM and $f : \mathbb{R}^{d+1} \to \mathbb{R}$ be such that the partial derivatives $\partial_i f$ and $\partial_{jk} f$ exist for all $1 \leq i \leq d+1, 1 \leq j, k \leq d$ and are continuous. If for some $t > 0$

$$E\left[\int_0^t |\nabla_x f(B_s, s)|^2 ds\right] < \infty$$

where

$$\nabla_x f = (\partial_1 f, ..., \partial_d f)$$

then, almost surely, for all $0 \leq s \leq t$

$$f(B_s, s) - f(B_0, 0) = \int_0^s \nabla_x f(B_u, u) dB_u + \int_0^s \partial_{d+1} f(B_u, u) du + \frac{1}{2}\int_0^s \triangle_x f(B_u, u) du$$

where $\triangle_x f = \sum_{j=1}^d \partial_{jj} f$.

7 Pathwise stochastic integration with respect to continuous semimartingales

We saw that for a continuous process H with $E[\int_0^t H_s^2 ds] < \infty$, $\int_0^t H_s dB_s$ can be defined as an almost sure limit, see Remark 5.7.
Can we replace $(B_t)_{t\geq 0}$ with another continuous martingale?

Definition 7.1 Let $E_n = \{0 = t_1^{(n)} < ... < t_n^{(n)}\}$ be a sequence of partitions with $s(E_n) = \sup |t_{i+1}^{(n)} - t_i^{(n)}| \to 0$.
Then, the function X_t has **continuous quadratic variation** along the sequence E_n if

$$\langle X\rangle_t = \lim_{n\to\infty} \sum_{\substack{t_i \in E_n \\ t_{i+1} \leq t}} (X_{t_{j+1}} - X_{t_j})^2 \tag{7.1}$$

exists P-a.s.

Remark 7.2 $\langle X\rangle_t$ is increasing and continuous, hence it defines a measure ν on $(\mathbb{R}, \mathcal{B})$ given by $\nu((a, b]) = \langle X\rangle_b - \langle X\rangle_a$.
In particular, if $f : \mathbb{R} \to \mathbb{R}$ is continuous, $\int_0^t f(s) d\langle X\rangle_s$ is well-defined.

In analogy to Theorem 6.4, we have

Theorem 7.3 *Assume that X_t has continuous quadratic variation $\langle X\rangle_t$ and $f:\mathbb{R}\to\mathbb{R}$ is continuous. Then, for $t>0$,*

$$\sum_{\substack{t_i\in E_n\\ t_{i+1}\le t}} f(X_{t_i})(X_{t_{j+1}}-X_{t_j})^2 \to \int_0^t f(X_s)d\langle X\rangle_s$$

Proof: Clear for $f\equiv 1$ due to (7.1). Rest see literature.
Note that the assumption "X_t has continuous quadratic variation" can only be satisfied for continuous processes.

Theorem 7.4 (Itô's formula for (deterministic) functions with continuous quadratic variation)
Assume that the function X_t has continuous quadratic variation $\langle X\rangle_t$ and let $f:\mathbb{R}\to\mathbb{R}$ be twice continuous differentiable. Then,

$$f(X_t)-f(X_0)=\int_0^t f'(X_u)dX_u+\frac{1}{2}\int_0^t f''(X_u)d\langle X\rangle_u \tag{7.2}$$

where

$$\int_0^t f'(X_u)dX_u=\lim_{n\to\infty}\sum_{\substack{t_i\in E_n\\ t_{i+1}\le t}} f'(X_{t_i})(X_{t_{i+1}}-X_{t_i}).$$

Sketch of proof: As in the proof of Theorem 6.1, we have

$$\Big|\sum_{\substack{t_i\in E_n\\ t_{i+1}\le t}} f(X_{t_{i+1}})-f(X_{t_i})-\sum_{\substack{t_i\in E_n\\ t_{i+1}\le t}} f'(X_{t_i})(X_{t_{i+1}}-X_{t_i})-\sum_{\substack{t_i\in E_n\\ t_{i+1}\le t}}\frac{1}{2}f'(X_{t_i})(X_{t_{i+1}}-X_{t_i})^2\Big|$$

$$\le w(\delta_X,M_X)\sum_{\substack{t_i\in E_n\\ t_{i+1}\le t}}(X_{t_{i+1}}-X_{t_i})^2$$

where

$$\delta_X=\max_{\substack{t_i\in E_n\\ t_{i+1}\le t}}|X_{t_{i+1}}-X_{t_i}|$$

and

$$M_X=\max_{0\le s\le t}|X_s|$$

and

$$w(\delta,M)=\sup_{\substack{s,t\in[-M,M]\\ |s-t|<\delta}}|f''(s)-f''(t)|.$$

Now,

$$\sum_{\substack{t_i\in E_n\\ t_{i+1}\le t}}(f(X_{t_{i+1}})-f(X_{t_i}))\to f(X_t)-f(X_0)$$

and

$$\sum_{\substack{t_i\in E_n\\ t_{i+1}\le t}} f''(X_{t_i})(X_{t_{i+1}}-X_{t_i})^2\to\int_0^t f''(X_s)d\langle X\rangle_s$$

and

$$\sum_{\substack{t_i \in E_n \\ t_{i+1} \le t}} (X_{t_{i+1}} - X_{t_i})^2 \to \langle X \rangle_t.$$

Moreover, $w(\delta_X, M_X) \underset{\delta \to 0}{\longrightarrow} 0$. Hence,

$$\sum_{\substack{t_i \in E_n \\ t_{i+1} \le t}} f'(X_{t_i})(X_{t_{i+1}} - X_{t_i})$$

has to converge as well and (7.2) holds.

Remark 7.5 (1) Theorem 7.4 gives a "pathwise" version of Itô's formula, "without probability".

(2) If X is BV on $[0,t]$, $\langle X \rangle_t = 0$ and (7.2) becomes

$$f(X_t) - f(X_0) = \int_0^t f'(X_u) dX_u.$$

Short notation:

$$df(X) = f'(X)dX \quad \text{“classical differential”}$$

If X has continuous quadratic variation $\langle X \rangle$ and $\langle X \rangle_t \neq 0$, then

$$df(X) = f'(X)dX + \frac{1}{2} f''(X) d\langle X \rangle \quad \text{“Itô differential”}$$

We have defined $\int_0^t f'(X_u)dX_u$ for all continuous X_t with continuous quadratic variation $\langle X \rangle_t$. Which stochastic processes X_t have a.s. continuous quadratic variation $\langle X \rangle_t$?

Lemma 7.6 *(i) Assume X_t has continuous quadratic variation $\langle X \rangle_t$. Then, if f is continuously differentable, $f(X_t)$ has continuous quadratic variation*

$$\langle f(X) \rangle_t = \int_0^t f'(X_s)^2 d\langle X \rangle_s$$

(ii) Let $X_t = M_t + A_t, t \ge 0$, where M_t has quadratic variation $\langle M \rangle_t$ and A_t has quadratic variation $\langle A \rangle_t = 0$, then X_t has quadratic variation $\langle X \rangle_t$ and $\langle X \rangle_t = \langle M \rangle_t$.

(iii) Let $f \in C^1(\mathbb{R})$ and assume X_t has continuous quadratic variation $\langle X \rangle_t$. Then, $M_t = \int_0^t f(X_s)dX_s$ has continuous quadratic variation

$$\langle M \rangle_t = \int_0^t f(X_s)^2 d\langle X \rangle_s.$$

(iv) Let $f \in C^1(\mathbb{R}^2)$ and assume X_t has continuous quadratic variation $\langle X \rangle_t$. Then, $g(t) = f(X_t, t)$ has continuous quadratic variation

$$\langle g \rangle_t = \int_0^t \left(\frac{\partial f}{\partial x}(X_s, s) \right)^2 d\langle X \rangle_s.$$

Examples:

1. $(B_t)_{t\geq 0}$ BM, $\alpha > 0, Z_t = e^{\alpha B_t}, t \geq 0$. Then, Lemma 7.4 (i) implies that $\langle Z\rangle_t = \int_0^t \alpha^2 e^{2\alpha B_s} ds = \int_0^t \alpha^2 Z_s^2 ds$ P-a.s. Note that $\langle Z\rangle_t$ is random (whereas $\langle B\rangle_t = t, \forall t$ P-a.s.).
2. $(B_t)_{t\geq 0}$ BM, $\alpha > 0, M_t = e^{\alpha B_t - \frac{1}{2}\alpha^2 t}, t \geq 0$. Then, we know that $M_t = 1 + \int_0^t \alpha M_s dB_s, t \geq 0$. Hence, Lemma 7.4 (iv) implies that $\langle M\rangle_t = \int_0^t \alpha^2 M_s^2 ds$ P-a.s. Note that $\langle M\rangle_t$ is random (whereas $\langle B\rangle_t = t, \forall t$ P-a.s.).

Proof of Lemma 7.4:

(i) $\triangle_i X := X_{t_{i+1}} - X_{t_i}$. Then

$$f(X_{t_{i+1}}) - f(X_{t_i}) = f(X_{t_i})\triangle_i X + R_i$$

and

$$|R_i| \leq \sup_{s,u\in[t_i,t_{i+1}]} |f'(X_s) - f'(X_u)| \cdot \triangle_i X$$

Hence,

$$\sum_{\substack{t_i\in E_n\\ t_{i+1}\leq t}} (f(X_{t_{i+1}}) - f(X_{t_i}))^2 = \sum_{\substack{t_i\in E_n\\ t_{i+1}\leq t}} f'(X_{t_i})^2(\triangle_i X)^2 + \sum_{\substack{t_i\in E_n\\ t_{i+1}\leq t}} R_i^2 + 2\sum_{\substack{t_i\in E_n\\ t_{i+1}\leq t}} f'(X_{t_i})\triangle_i X \cdot R_i$$

But

$$\sum_{\substack{t_i\in E_n\\ t_{i+1}\leq t}} f'(X_{t_i})^2(\triangle_i X)^2 \to \int_0^t f'(X_s)^2 d\langle X\rangle_s$$

due to Theorem 7.3.
Rest: Exercise.

(ii) $\triangle_i M = M_{t_{i+1}} - M_{t_i}$, $\triangle_i A = A_{t_{i+1}} - A_{t_i}$. Then

$$(\triangle_i X)^2 = (\triangle_i M)^2 + (\triangle_i A)^2 + 2\triangle_i M\triangle_i A.$$

Hence

$$\sum_{\substack{i:t_i\in E_n\\ t_{i+1}\leq t}} (\triangle_i X)^2 = \sum_{\substack{i:t_i\in E_n\\ t_{i+1}\leq t}} (\triangle_i M)^2 + \sum_{\substack{i:t_i\in E_n\\ t_{i+1}\leq t}} (\triangle_i A)^2 + 2\sum_{\substack{i:t_i\in E_n\\ t_{i+1}\leq t}} \triangle_i M\triangle_i A \to \langle M\rangle_t,$$

since

$$\sum_{\substack{i:t_i\in E_n\\ t_{i+1}\leq t}} (\triangle_i A)^2 \to 0$$

and

$$\sum_{\substack{i:t_i\in E_n\\ t_{i+1}\leq t}} \triangle_i M\triangle_i A \to 0$$

(details: exercise).

(iii) Due to (7.2), taking g such that $g' = f$,

$$M_t = g(X_t) - g(X_0) - \frac{1}{2}\int_0^t g''(X_s)d\langle X\rangle_s.$$

But $\int_0^t g''(X_s)d\langle X\rangle_s$ is continuous and BV as a function of t. Using (ii) and (i),

$$\langle M\rangle_t = \langle g(X)\rangle_t = \int_0^t g'(X_s)^2 d\langle X\rangle_s = \int_0^t f(X_s)^2 d\langle X\rangle_s$$

(iv) See [5], Remark 1.3.22.

☻

Theorem 7.7 *If $(X_t)_{t\geq 0}$ is a continuous martingale with $X_0 = 0$ and $E[X_t^2] < \infty$, $\forall t$ there exists a unique process $\langle X\rangle_t$ with $\langle X\rangle_0 = 0$ which is continuous, adapted and increasing, such $X_t^2 - \langle X\rangle_t$ is a martingale. Moreover, X_t has quadratic variation $\langle X\rangle_t$.*

Proof: If $(B_t)_{t\geq 0}$ is a BM and $(H_t)_{t\geq 0}$ progressively measurable and $X_t = \int_0^t H_s dB_s$, then $\langle X\rangle_t = \int_0^t H_s^2 ds$ P-a.s. (see Lemma 5.4 and Lemma 7.4(i).)
General case: See literature. ☻

$\langle X\rangle_t$ is called **quadratic variation** or **bracket** or **compensator** of (X_t).
The first part of Theorem 7.6 has a discrete time analogue.
A stochastic process (Y_n) is **previsible** w.r.t. a filtration $(\mathcal{A}_n)$ if Y_n is $\mathcal{A}_{n-1}$-measurable $\forall n$.

Theorem 7.8 (Doob decomposition)
Suppose $(X_n)_{n=0,1,2,...}$ is a martingale w.r.t. the filtration $(\mathcal{A}_n)$ and $E[X_n^2] < \infty$, $\forall n$. Then, there is a unique previsible increasing process (A_n) with $A_0 = 0$ such that $(X_n^2 - A_n)_{n=0,1,2,...}$ is a martingale w.r.t. $(\mathcal{A}_n)$.

Proof: Let $A_0 = 0$ and define, for $n \geq 1$, $A_n = A_{n-1} + E[X_n^2|\mathcal{A}_{n-1}] - X_{n-1}^2$. Clearly, A_n is $\mathcal{A}_{n-1}$-measurable $\forall n$. Since $(X_n^2)_{n=0,1,2,...}$ is a submartingale w.r.t. $\mathcal{A}_n$, A_n is increasing. Further, $E[X_n^2 - A_n|\mathcal{A}_{n-1}] = E[X_{n-1}^2 - A_{n-1}|\mathcal{A}_{n-1}] = X_{n-1}^2 - A_{n-1} \Rightarrow (X_n^2 - A_n)_{n=0,1,2,...}$ is a martingale w.r.t. $(\mathcal{A}_n)$.
To prove the uniqueness, assume that (A_n) and (B_n) both fulfill the requirements. Then $A_n - B_n$ is a previsible martingale starting at 0, and we infer that $A_n - B_n = E[A_n - B_n|\mathcal{A}_{n-1}]$, hence, by induction, $A_n = B_n$ $\forall n$. ☻

Definition 7.9 A continuous semimartingale w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ is a process $(X_t)_{t\geq 0}$ which is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and which has a decomposition $X_t = X_0 + M_t + A_t$, $\forall t \geq 0$ P-a.s. where $(M_t)_{t\geq 0}$ is a continuous martingale and $(A_t)_{t\geq 0}$ is a continuous adapted process which is BV (on each interval $[0,t]$).
If $E[M_t^2] < \infty$, $\forall t$, we define for $f \in C^1(\mathbb{R})$

$$\int_0^t f(s)dX_s := \int_0^t f(s)dM_s + \int_0^t f(s)dA_s$$

8 Cross-variation and Itô's product rule

Definition 8.1 The cross-variation $\langle X, Y \rangle$ is given by

$$\langle X, Y \rangle_t = \lim_{n\to\infty} \sum_{\substack{t_i \in E_n \\ t_{i+1} \le t}} (X_{t_{i+}} - X_{t_i})(Y_{t_{i+}} - Y_{t_i})$$

(provided that the limit exists).
Clearly, $\langle X, X \rangle = \langle X \rangle$.

Lemma 8.2 *The following statements are equivalent*

(i) $\langle X, Y \rangle$ *exists and* $t \to \langle X, Y \rangle_t$ *is continuous.*

(ii) $\langle X + Y \rangle$ *exists and* $t \to \langle X + Y \rangle_t$ *is continuous.*

If (i) and (ii) hold, then

$$\langle X, Y \rangle = \frac{1}{2}(\langle X + Y \rangle - \langle X \rangle - \langle Y \rangle) \tag{8.1}$$

In particular, in this case $\int_0^t g(s) d\langle X, Y \rangle_s$ *is well-defined for* $g \in C[0, \infty)$.

Proof:

$$(X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) =$$
$$\frac{1}{2}\left(\left((X_{t_{i+1}} + Y_{t_{i+1}}) - (X_{t_i} + Y_{t_i}) \right)^2 - (X_{t_{i+1}} - X_{t_i})^2 - (Y_{t_{i+1}} - Y_{t_i})^2 \right).$$

☻

Example 8.3 Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be **independent** BMs.
Claim:
For P-a.a. ω

$$\langle X(\omega), Y(\omega) \rangle_t = 0 \quad \forall t$$

Proof:

$$\langle X, Y \rangle = \frac{1}{2}(\langle X + Y \rangle - \langle X \rangle - \langle Y \rangle)$$

and we have $\langle X \rangle_t = t$, $\forall t$, and $\langle Y \rangle_t = t$, $\forall t$ P-a.s. Suffices to show $\langle X + Y \rangle = 2t$ $\forall t$ P-a.s. $Z_t := \frac{1}{\sqrt{2}}(X_t + Y_t)$, $t \geq 0$ is again a BM (Proof: exercise). Therefore, $\langle Z \rangle_t = t$ $\forall t$ P-a.s. implying that $\langle X + Y \rangle = 2t$, $\forall t$ P-a.s. ☻

Lemma 8.4 X, $\langle X \rangle$ *continuous as before,* $f, g \in C^1(\mathbb{R})$,

$$Y_t = \int_0^t f(X_s) dX_s, \qquad Z_t = \int_0^t g(X_s) dX_s$$

Then

$$\langle Y, Z \rangle_t = \int_0^t f(X_s) g(X_s) d\langle X \rangle_s \tag{8.2}$$

Proof: Define F and G by $F' = f$, $G' = g$, $F(0) = G(0) = 0$.
Then, $Y_t = F(X_t) + A_t$ where

$$A_t = -F(X_0) - \frac{1}{2}\int_0^t F''(X_s)d\langle X\rangle_s$$

and $Z_t = G(X_t) - B_t$ where

$$B_t = -G(X_0) - \frac{1}{2}\int_0^t G''(X_s)d\langle X\rangle_s.$$

A_t, B_t are continuous with quadratic variation 0. Hence,

$$\begin{aligned}\langle Y, Z\rangle_t &= \frac{1}{2}(\langle Y+Z\rangle_t - \langle Y\rangle_t - \langle Z\rangle_t)\\ &= \frac{1}{2}\left(\int_0^t (f(X_s)+g(X_s))^2 d\langle X\rangle_s - \int_0^t f(X_s)^2 d\langle X\rangle_s - \int_0^t g(X_s)^2 d\langle X\rangle_s\right)\\ &= \int_0^t f(X_s)g(X_s)d\langle X\rangle_s\end{aligned}$$

☻

Theorem 8.5 (Itô's formula in d dimensions)
$X = (X^{(1)}, ..., X^{(d)}) : [0,\infty) \to \mathbb{R}^d$ where $X^{(i)}$, $1 \le i \le d$ are continuous with continuous quadratic variation $\langle X^{(i)}\rangle_t$ and continuous cross-variations $\langle X^{(i)}, X^{(k)}\rangle_t$, $1 \le i,\ k \le d$. Let $f \in C^2(\mathbb{R}^d)$. Then,

$$f(X_t) - f(X_0) = \int_0^t (\nabla f, dX_s) + \frac{1}{2}\int_0^t \sum_{i,k=1}^d \frac{\partial^2 f}{\partial x_i \partial x_k}(X_s)d\langle X^{(i)}, X^{(k)}\rangle_s$$

where

$$\int_0^t (\nabla f, dX_s) = \sum_{k=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s)dX_s^{(i)}.$$

Proof: Analogous to the proof of Theorem 6.1: Taylor formula for$f(X_{t_{i+1}}) - f(X_{t_i})$. See literature. ☻
Note that Theorem 6.8 follows, using Example 8.3.

Corollary 8.6 (Itô's product rule)
Assume that $X, Y, \langle X\rangle, \langle Y\rangle$ and $\langle X, Y\rangle$ are continuous. Then, $\forall t > 0$

$$X_t \cdot Y_t = X_0 \cdot Y_0 + \int_0^t Y_s dX_s + \int_0^t X_s dY_s + \langle X, Y\rangle_t \tag{8.3}$$

Short notation:

$$d(X \cdot Y) = Y dX + X dY + d\langle X, Y\rangle \tag{8.4}$$

Proof: Apply Theorem 8.5 with $d = 2, f(x, y) = x \cdot y$. ☻

Example 8.7 (Ornstein-Uhlenbeck process)
Let $\alpha > 0$ and $(B_t)_{t\geq 0}$ a BM, $x_0 \in \mathbb{R}$. Then, we say that

$$X_t = e^{-\alpha t}x_0 + e^{-\alpha t}\int_0^t e^{\alpha s}dB_s, \qquad t \geq 0 \tag{8.5}$$

is an Ornstein-Uhlenbeck process with parameter α and starting point x_0.
Claim:
$(X_t)_{t\geq 0}$ solves the stochastic differential equation

$$\begin{cases} dX_t = dB_t - \alpha X_t dt \\ X_0 = x_0 \end{cases} \quad i.e. \begin{cases} X_t = B_t - \int_0^t \alpha X_s ds \\ X_0 = x_0 \end{cases}$$

Proof:

$$\begin{aligned} X_t &= e^{-\alpha t}x_0 + e^{-\alpha t}\int_0^t e^{\alpha s}dB_s \\ dX_t &= -\alpha x_0 e^{-\alpha t}dt - \alpha(e^{-\alpha t}\int_0^t e^{\alpha s}dB_s)dt + e^{-\alpha t}e^{\alpha t}dB_t \\ &= -\alpha X_t dt + dB_t\,, \end{aligned}$$

where we applied Itô's product rule to $e^{-\alpha t}\int_0^t e^{\alpha s}dB_s$. ☻
Claim: If $X_0 \stackrel{d}{=} N(0, \frac{1}{2\alpha})$, X_0 independent of $(B_t)_{t\geq 0}$, then

$$X_t = e^{-\alpha t}X_0 + e^{-\alpha t}\int_0^t e^{\alpha s}dB_s$$

is a Gaussian process with $E[X_t] = 0, \forall t$ and $\mathrm{Cov}(X_s, X_t) = \frac{1}{2\alpha}e^{-\alpha|t-s|}$. Hence, for any t, $E[X_t^2] = \frac{1}{2\alpha} \Rightarrow X_t \stackrel{d}{=} N(0, \frac{1}{2\alpha})$.
In particular, for the process $Z_t = e^{-t}B_{e^{2t}}, t \geq 0$, in Example 2.4, $(\frac{1}{\sqrt{2}}Z_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process with $\alpha = 1$ and $X_0 = N(0, \frac{1}{2})$.

Proof: Clearly, $E[X_t] = 0, \forall t$. Assume $s \leq t$. Then,

$$E[X_s X_t] = E[X_0^2]e^{-\alpha t}e^{-\alpha s} + 0 + 0 + e^{-\alpha(s+t)}E[\int_0^t e^{\alpha u}dB_u \int_0^s e^{\alpha v}dB_v]$$

For any martingale $(M_t)_{t\geq 0}$, $E[M_t M_s] = E[M_s^2]$ for $s \leq t$.(Proof: exercise).
Hence, applying this to the martingale$M_t = \int_0^t e^{\alpha u}dB_u$,

$$\begin{aligned} E[X_s X_t] &= \frac{1}{2\alpha}e^{-\alpha(t+s)} + e^{-\alpha(t+s)}E[(\int_0^s e^{\alpha v}dB_v)^2] \\ &= \frac{1}{2\alpha}e^{-\alpha(t+s)} + e^{-\alpha(t+s)}\int_0^s e^{2\alpha v}dv \\ &= \frac{1}{2\alpha}e^{-\alpha(t+s)} + e^{-\alpha(t+s)}\frac{1}{2\alpha}(e^{2\alpha s} - 1) \\ &= \frac{1}{2\alpha}e^{-\alpha(t-s)}\,. \end{aligned}$$

☻

9 Stochastic Differential Equations

In this chapter we want to study stochastic differential equations (SDE) of the form:

$$X_0 = \xi$$

$$dX_t = \sigma(t, X_t) \cdot dB_t + b(t, X_t)dt \tag{9.1}$$

$(X_t)_{t\geq 0} = (X_t^{(1)}, X_t^{(2)}, ..., X_t^{(n)})_{t\geq 0}$ is an unknown $\mathbb{R}$-valued process,
$(B_t)_{t\geq 0} = (B_t^{(1)}, B_t^{(2)}, ..., B_t^{(m)})_{t\geq 0}$ is an m-dimensional Brownian motion and $b(t, x)$ and $\sigma(t, x)$ are measurable functions of $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$. The drift vector $b(t, x)$ is $\mathbb{R}^n$-valued and the dispersion matrix $\sigma(t, x)$ is an $n \times m$-matrix valued. Further ξ is a random variable with values in $\mathbb{R}^n$ which is independent of $(B_t)_{t\geq 0}$.

Definition 9.1 We say the SDE (9.1) has the strong solution $(X_t)_{t\geq 0}$ if the following conditions hold:

(i) $(X_t)_{t\geq 0}$ is adapted to the filtration $(\mathcal{F}_t^\xi)_{t\geq 0}$ where $\mathcal{F}_t^\xi$ is the completion of $\sigma(\xi, \{B_s : s \leq t\})$ for $t \geq 0$.

(ii) $(X_t)_{t\geq 0}$ satisfies the following integral equation:

$$X_t^{(i)} = \xi^{(i)} + \sum_{j=1}^{m} \int_0^t \sigma_{ij} dB_s^{(j)} + \int_0^t b_i(s, X_s)ds \tag{9.2}$$

for $t \geq 0, i = 1, 2, ...n$.

Remark 9.2 (1) Processes which fulfill the integral equations in (9.2) and are defined on a possibly enlarged probability space, but do not have to be adapted to the filtration $(\mathcal{F}_t^\xi)_{t\geq 0}$, are called **weak solutions**.

(2) The second property of a solution of a SDE includes the requirement that the integrals in (9.2) are well–defined.

Example 9.3 For $\alpha, \beta \in \mathbb{R}$ consider the SDE

$$X_0 = 1$$

$$dX_t = \alpha X_t dB_t + \beta X_t dt$$

for a one-dimensional BM $(B_t)_{t\geq 0}$. This SDE has the (unique) strong solution $X_t = \exp(\alpha B_t + (\beta - \frac{\alpha^2}{2})t)$.

Proof: exercise

Remark
The SDE from Example 9.3 is used for the Black-Scholes model.

Remark 9.4 In the following Theorem we use

$$||x||^2 = \sum_{i=1}^{n} (x_i)^2 \qquad \text{for } x = (x_1, ..., x_n) \in \mathbb{R}^n \,,$$

$$||\sigma||^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} (\sigma_{ij})^2 \qquad \text{for } \sigma = (\sigma_{ij})_{ij} \in \mathbb{R}^{n \times m} \,.$$

Theorem 9.5 (Existence and uniqueness)
Assume that $b : \mathbb{R}_+ \times \mathbb{R}^n$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable and satisfy the Lipschitz-condition

$$||\sigma(t,x) - \sigma(t,y)|| + ||b(t,x) - b(t,y)|| \leq K \cdot ||x - y|| \tag{9.3}$$

as well as the growth condition

$$||\sigma(t,x)||^2 + ||b(t,x)||^2 \leq K^2 \cdot (1 + ||x||^2) \tag{9.4}$$

for a constant $K > 0$, for all $t \geq 0$ and $x, y \in \mathbb{R}^n$. Let ξ be a random variable with values in $\mathbb{R}^n$ which is independent of the m-dimensional BM $(B_t)_{t\geq 0}$ such that $E[||\xi||^2] < \infty$. Then the SDE (9.1) has a unique, continuous, strong solution $(X_t)_{t\geq 0}$ such that

$$E\left[\int_0^T ||X_t||^2 dt\right] < \infty \qquad \forall T > 0 \tag{9.5}$$

Remark 9.6 Uniqueness in Theorem 9.5 means that for two continuous solutions $(X_t)_{t\geq 0}$, $(X'_t)_{t\geq 0}$ of the SDE (9.1) which fulfill the properties of Theorem 9.5 we have

$$P(X_t = X'_t \quad \forall t \geq 0) = 1.$$

Example 9.7 To illustrate that we need some conditions like (9.3) and (9.4) let us look at the following examples from ODEs:

$$\frac{dX_t}{dt} = (X_t)^2, \quad X_0 = 1$$

corresponding to $b(x) = x^2$ (which does not satisfy (9.4)) has the unique solution

$$X_t = \frac{1}{1-t} \quad \text{for } 0 \leq t < 1.$$

Another example:

$$\frac{dX_t}{dt} = 3(X_t)^{2/3}, \quad X_0 = 0$$

where $b(x) = 3x^{2/3}$ does not satisfy (9.3) at $x = 0$ and

$$X_t = \begin{cases} 0, & \text{for } t \leq a, \\ (t-a)^3, & \text{for } t > a. \end{cases}$$

are solutions for all $a > 0$.

Sketch of the proof of Theorem 9.5:

Uniqueness:

Uses the following lemma

Lemma 9.8 *Gronwall inequality*

Let $f : [0, T] \to \mathbb{R}$ *be integrable and* $A \in \mathbb{R}, C > 0$ *such that*

$$f(t) \leq A + C \cdot \int_0^t f(s)ds \qquad \forall t \in [0, T],$$

Then

$$f(t) \leq A + e^{Ct} \qquad \forall t \in [0, T] \,.$$

Proof: see literature.

Consider two continuous solutions $(X_t)_{t\geq 0}, (X_t')_{t\geq 0}$ of the SDE (9.1) which fulfill condition (9.5)

$$X_t - X_t' = \int_0^t (\sigma(s, X_s) - \sigma(s, X_s')) \cdot dB_s + \int_0^t (b(s, X_s) - b(s, X_s'))ds$$

Therefore (using $(a + b)^2 \leq 2a^2 + 2b^2$),

$$||X_t - X_t'|| \leq 2|| \int_0^t (\sigma(s, X_s) - \sigma(s, X_s')) \cdot dB_s||^2 + 2|| \int_0^t (b(s, X_s) - b(s, X_s'))ds||^2$$

A short calculation for the first term (see Theorem 5.6 for $n = m = 1$) and an application of the Cauchy-Schwarz inequality shows

$$E[||X_t - X_t'||^2] \leq 2 \int_0^t E[||(\sigma(s, X_s) - \sigma(s, X_s'))||^2]dB_s + 2t \int_0^t E[||(b(s, X_s) - b(s, X_s'))||^2]ds$$

Therefore we have for $f(t) := E[||X_t - X_t'||^2]$ and $C := 2(T + 1)K^2$ (for $T > 0$) due to condition (9.3)

$$f(t) \leq C \cdot \int_0^t f(s)ds \qquad \text{for } 0 \leq t \leq T$$

and the Gronwall inequality implies $f \equiv 0$. We can conclude due to the continuity of $(X_t)_{t\geq 0}, (X_t')_{t\geq 0}$ that we have

$$P(X_t = X_t' \quad \forall t \geq 0) = 1.$$

(using the fact that $\mathbb{Q}_+$ is countable and dense in $\mathbb{R}_+$).

Existence: We define for $n \in \mathbb{N}$

$$X_t^{(n+1)} := \xi + \int_0^t \sigma(s, X_s^{(n)}) \cdot dB_s + \int_0^t b(s, X_s^{(n)})ds \qquad \text{for } t \geq 0$$

inductively where $X_t^0 := \xi$. Using the growth condition we can show inductively that for $T > 0$

$$\int_0^T E[||X_s^{(n)}||^2]ds < \infty \,,$$

i.e. the stochastic integral is well–defined in every step of the recursion. One can show that $(X^{(n)})_{n\in\mathbb{N}_0}$ converges a.s. uniformly on $[0, T]$ for every $T > 0$ and the limit solves the SDE (9.1) and has the properties of Theorem 9.5 (more details can be found in the literature). ☻

10 Girsanov transforms

<u>Goal:</u>
Construction of a stochastic process $(X_t)_{t\geq 0}$ with

$$dX_t = b(X_t, t)dt + dB_t$$

$$X_0 = x_0$$

where $(B_t)_{t\geq 0}$ is a BM, i.e.

$$X_t = x_0 + \int_0^t b(X_s, s)ds + B_t \tag{10.1}$$

Interpretation:
Deterministic process X_t with $\frac{d}{dt}X_t = b(X_t, t)$ with additional "noise" $(B_t)_{t\geq 0}$.
Possible strategies:

1) **Construction of a "strong" solution:**
For a given BM $(B_t)_{t\geq 0}$ on $(\Omega, \mathcal{A}, P)$, solve (10.1).

Example 10.1 (Ornstein-Uhlenbeck process)

$$dX_t = dB_t - \alpha X_t dt$$

$$X_0 = x_0$$

has the strong solution

$$X_t = e^{-\alpha t}(x_0 + \int_0^t e^{\alpha s} dB_s)\,, \tag{10.2}$$

see Example 8.8.

Drawback: a strong solution does not always exist.

2) **Construction of a "weak" solution:**
Find a BM $(B_t)_{t\geq 0}$ and a process $(X_t)_{t\geq 0}$ on some probability space $(\Omega, \mathcal{A}, P)$ such that (10.1) holds, i.e. find $(X_t)_{t\geq 0}$ such that

$$B_t = X_t - x_0 - \int_0^t b(X_s, s)ds$$

is a BM.

General Girsanov transformation
$(\Omega, \mathcal{A}, P)$ probability space, $(\mathcal{F}_t)_{t\geq 0}$ filtration, $\widetilde{P}$ probability measure on $(\Omega, \mathcal{A})$. Assume that for all t, $\widetilde{P}|_{\mathcal{F}_t} \ll P|_{\mathcal{F}_t}$.
Then, there are Radon-Nikodym derivatives

$$Z_t := \frac{d\widetilde{P}}{dP}|_{\mathcal{F}_t} = \frac{d\widetilde{P}|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}} \tag{10.3}$$

and $(Z_t)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ and P (see Probability Theory lecture notes, 14.1.3.)
We always assume that $(Z_t)_{t\geq 0}$ has continuous paths and that

$$\inf\{t \geq 0 : Z_t(\omega) = 0\} = \infty \qquad P\text{-a.s.}$$

Definition 10.2 $(M_t)_{t\geq 0}$ is a **local martingale** (up to ∞) if there is a sequence of stopping times $T_1 \leq T_2 \leq ...$ such that

i) $\sup_n T_n = \infty \qquad P$-a.s.

ii) $(M_{t\wedge T_n})_{t\geq 0}$ is a martingale, $\forall n$.

Each martingale is a local martingale but there are local martingales which are not martingales.
We will use the following important fact:
If M is a continuous local martingale and $(Y_t)_{t\geq 0}$ a continuous adapted process, then $\int_0^t Y_s dM_s$, $t \geq 0$ is again a continuous **local** martingale. See the literature, for instance [5], Proposition 1.4.29.

Lemma 10.3 *In the above setup, with* $Z_t := \frac{d\widetilde{P}}{dP}|_{\mathcal{F}_t}$*, the following holds.*

(i) For $s \leq t$ *and a function* g_t *which is* $\mathcal{F}_t$*-measurable and bounded, we have*

$$\widetilde{E}[g_t|\mathcal{F}_s] = \frac{1}{Z_s} E[g_t Z_t|\mathcal{F}_s] \qquad P\text{-a.s.}$$

where $\widetilde{E}$ *denotes expectation with respect to* $\widetilde{P}$*.*

(ii) For $\widetilde{M} = (\widetilde{M}_t)_{t\geq 0}$ *continuous and adapted, the following two statements are equivalent:*

a) $(\widetilde{M}_t)_{t\geq 0}$ *is a local martingale with respect to* $\widetilde{P}$*.*

b) $(\widetilde{M}_t Z_t)_{t\geq 0}$ *is a local martingale with respect to* P*.*

Proof:

(i) Assume g_s is $\mathcal{F}_s$-measurable and bounded.

$$\begin{aligned}
\widetilde{E}[g_s g_t] &= E[g_s g_t Z_t] \\
&= E[g_s E[g_t Z_t|\mathcal{F}_s]] \\
&= E\left[g_s Z_s E[g_t Z_t|\mathcal{F}_s]\frac{1}{Z_s}\right] \\
&= \widetilde{E}\left[g_s \frac{1}{Z_s} E[g_t Z_t|\mathcal{F}_s]\right]
\end{aligned}$$

$$\Rightarrow \widetilde{E}[g_t|\mathcal{F}_s] = \frac{1}{Z_s} E[g_t Z_t|\mathcal{F}_s]$$

(ii) Assume that $(\widetilde{M}_t Z_t)_{t\geq 0}$ is a martingale with respect to P. Then

$$\widetilde{E}[\widetilde{M}_t|\mathcal{F}_s] \underset{\text{(i)}}{=} \frac{1}{Z_s} E[\widetilde{M}_t Z_t|\mathcal{F}_s] = \frac{1}{Z_s}\widetilde{M}_s Z_s = \widetilde{M}_s, \text{ P-a.s.}$$

$\Rightarrow (\widetilde{M}_t)_{t\geq 0}$ is a martingale with respect to $\widetilde{P}$, hence b) $\Rightarrow$ a) for martingales.

Rest of the proof: Exercise. ☻

Theorem 10.4 *Assume that (M_t) is a continuous local martingale with respect to P and Z_t is defined as in (10.3). Then,*

$$\widetilde{M}_t = M_t - \int_0^t \frac{1}{Z_s} d\langle M, Z\rangle_s$$

is a continuous local martingale with respect to $\widetilde{P}$.
Short notation:

$$dM = d\widetilde{M} + \frac{1}{Z} d\langle M, Z\rangle$$

<u>Proof:</u> Due to Lemma 10.1, it suffices to show that $(\widetilde{M}_t Z_t)_{t\geq 0}$ is a continuous local martingale with respect to P.
Let $A_t = \int_0^t \frac{1}{Z_s} d\langle M, Z\rangle_s$. Then,

$$\widetilde{M}_t Z_t = Z_t(M_t - A_t) \overset{\text{Itô's product rule}}{=} Z_0 M_0 + \int_0^t (M_s - A_s) dZ_s - \int_0^t Z_s dA_s + \langle Z, M\rangle_t \quad (10.4)$$

The definition of A implies that

$$\int_0^t Z_s dA_s = \langle Z, M\rangle_t \,.$$

Due to (10.4), $\widetilde{M}_t Z_t - \widetilde{M}_0 Z_0$ is a stochastic integral, hence again a continuous local martingale with respect to P. ☻

Remark 10.5

$$\log Z_t = \log Z_0 + \int_0^t \frac{1}{Z_s} dZ_s - \int_0^t \frac{1}{Z_s^2} d\langle Z\rangle_s$$

Z continuous local martingale $\Rightarrow Y_t = \int_0^t \frac{1}{Z_s} dZ_s$ is a continuous local martingale with $\langle Y\rangle = \int_0^t \frac{1}{Z_s^2} d\langle Z\rangle_s$, hence

$$\log Z_t = \log Z_0 + Y_t - \frac{1}{2}\langle Y\rangle_t \Rightarrow Z_t = Z_0 e^{Y_t - \frac{1}{2}\langle Y\rangle_t}$$

and Z solves $dZ = Z dY$.
Using $\langle M, Z\rangle_t = \int_0^t Z_s d\langle M, Y\rangle_s$, see Lemma 10.6 below we get from Theorem 10.3 that

$$dM = d\widetilde{M} + d\langle M, Y\rangle \quad (10.5)$$

Lemma 10.6 (Generalization of Lemma 8.4)
Asume that M and Y are local martingales with continuous quadratic variation $\langle M\rangle$ and $\langle Y\rangle$ and $f, g \in C^2(\mathbb{R}^2)$. Let

$$V_t = \int_0^t f(M_s, s)dM_s, \qquad R_t = \int_0^t g(Y_s, s)dY_s\,.$$

Then,

$$\langle V, R\rangle_t = \int_0^t f(M_s, s)g(Y_s, s)d\langle M, Y\rangle_s.$$

Proof: See literature. For $M = Y$, we recover Lemma 8.4.
Now, consider

$$Z_t = \int_0^t Z_s dY_s, \qquad M_t = \int_0^t 1 dM_s$$

Lemma 10.6 yields

$$\langle M, Z\rangle_t = \int_0^t Z_s d\langle M, Y\rangle_s.$$

Girsanov transform on Wiener space
Let $(X_t)_{0\le t\le 1}$ be a BM with respect to P.

Theorem 10.7 (Girsanov)
Assume $(b_t)_{0\le t\le 1}$ is progressively measurable and adapted and $E[\int_0^1 b_s^2 ds] < \infty$. Let $Z_1 := e^{\int_0^1 b_s dX_s - \frac{1}{2}\int_0^1 b_s^2 ds}$ and assume

$$E[Z_1] = 1 \tag{10.6}$$

Define $\widetilde{P}$ by $\frac{d\widetilde{P}}{dP}|_{\mathcal{F}_1} = Z_1$.
Then, $B_t = X_t - \int_0^t b_s ds, 0 \le t \le 1$ is a BM with respect to $\widetilde{P}$.

For the proof, we will need

Theorem 10.8 (Lévy's characterization of BM)
Assume (B_t) is a continuous local martingale with quadratic variation $\langle B\rangle_t = t$. Then, $(B_t)_{t\ge 0}$ is a BM.

Proof: We use characteristic functions:
A probability measure μ on $\mathbb{R}$ is characterized by its Fourier transform:

$$\widehat{\mu}(u) = \int e^{iux}\mu(dx)\,.$$

(Example: $\mu = N(0, \sigma^2), \widehat{\mu}(u) = e^{-\frac{1}{2}\sigma^2 u^2}$).
We show that $B_t - B_s$ is independent of $\mathcal{F}_s$, with law $N(0, t-s)$. Itô's formula for $f(x) = e^{iux}$ (verify by separating real and imaginary parts) yields

$$e^{iuX_t} - e^{iuX_s} = \int_s^t iue^{iuX_r}dX_r + \frac{1}{2}\int_s^t -u^2 e^{iuX_r}dr$$

Divide by e^{iuX_s}, take conditional expectation w.r.t $\mathcal{F}_s$

$$\Rightarrow E[e^{iu(X_t-X_s)}|\mathcal{F}_s] - 1 = E\left[\int_s^t iue^{iu(X_r-X_s)dX_r}|\mathcal{F}_s\right] - \frac{1}{2}E\left[\int_s^t u^2e^{iu(X_r-X_s)}dr|\mathcal{F}_s\right]$$

Take $A \in \mathcal{F}_s$. Then,

$$E[e^{iu(X_t-X_s)}I_A] - P[A] = -\frac{1}{2}u^2\int_s^t E[e^{iu(X_r-X_s)}I_A]dr$$

Let $g(t) := E[e^{iu(X_t-X_s)}I_A]$. Then, $g(t) - g(s) = -\frac{1}{2}u^2\int\limits_s^t g(r)dr$ which implies $g(t) = g(s)\exp(-\frac{1}{2}u^2(t-s))$. Hence $E[e^{iu(X_t-X_s)}I_A] = P[A]e^{-\frac{1}{2}u^2(t-s)} = P[A]\widehat{\mu}(u)$ where $\mu = N(0, t-s)$, $\forall A \in \mathcal{F}_s$. This implies that $X_t - X_s$ is independent of $\mathcal{F}_s$ with law $N(0, t-s)$. ☻

Proof of Thm. 10.7: We have $\frac{d\widetilde{P}}{dP}|_{\mathcal{F}_t} = Z_t = e^{\int_0^t b_s dX_s - \frac{1}{2}\int_0^t b_s^2 ds}$ hence $Z_t = e^{Y_t - \frac{1}{2}\langle Y\rangle_t}$ with $Y_t = \int_0^t b_s dX_s$. Due to (10.4), $dX_t = dB_t + d\langle X, \int_0^\cdot b_s dX_s\rangle_t$ where $(B_t)_{t\geq 0}$ is a continuous local martingale w.r.t. $\widetilde{P}$.
Since $\langle X, \int_0^\cdot b_s dX_s\rangle_t = \int_0^t b_s ds$ (Lemma 8.4) we have $dX_t = dB_t + b_t dt$. Hence, P-a.s. $\langle B_t\rangle = t$, $X_t = B_t + \int_0^t b_s ds$. $\Rightarrow$ $\widetilde{P}$-a.s. $\langle B\rangle_t = t$, $\forall t$ since $\widetilde{P} \ll P$.
Hence, Lévy's characterization of BM implies that $(B_t)_{t\geq 0}$ is a BM w.r.t. $\widetilde{P}$. ☻

Remark 10.9 Thm. 10.5 also applies to $b_t = b(X_u, u \leq t)$ and yields a weak solution of $dX_t = dB_t - b_t dt$, i.e. b_t can depend on the whole "history" up to time t. (10.5) and (10.6) are only weak regularity assumptions.

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References

[1] Ioannis Karatzas and Steven Shreve: Brownian motion and stochastic calculus, Springer, 2007.

[2] Achim Klenke: Probability Theory, Springer, 2008.

[3] Thomas Liggett: Continuous time Markov processes: An introduction, American Mathematical Society, 2010.

[4] Peter Mörters and Yuval Peres: Brownian motion, Cambridge University Press, 2010.

[5] Michael Röckner: Introduction to stochastic analysis. Available online at http://www.math.uni-bielefeld.de/∼roeckner/teaching1112/stochana.pdf